# SOME DISCRETE REAL AND COMPLEX FOURIER TRANSFORMS, A DISCUSSION, WITH EXAMPLES

**BY**

**CHRIS ANGOVE**

**EQUATION SECTION (NEXT)**

January 2009

## PREFACE

Having done quite a bit on RF I decided it was time to improve my knowledge on the DSP side of things so where better to start than the Fourier transforms. Their use in communication systems is of course just one application, but a very important one. For digital communications in general, including a very good presentation of FTs, I can recommend a very good book by Bernard Sklar [1]. As good engineering reference books go, it was relatively cheap. I paid about GBP 42 (for the 1000+ pages hardback) version of it in early 2008. It has a very thorough treatment of most aspects of digital communications and explains things well. The 'Smith family' seem to be well represented in DSP and digital communications books with David R. Smith [2] and Steven W. Smith [3]. The latter has some very useful information which may be downloaded from their website http://www.thedspguide.com including an electronic copy of the book.

To try out some of the theory by way of simulations you don't need to spend a lot of money on mathematics applications like MathCAD: in many cases you can actually use Microsoft Excel, but you must have the so-called engineering functions enabled from the Analysis ToolPak add-in to provide a range of complex functions. They are a bit verbose and you may wish to test some of them on some simple complex numbers to get used to the syntax. Then you can do some of the integrations iteratively on the spreadsheets. I have included some examples of these.

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# 1 THE DISCRETE REAL FOURIER TRANSFORM

One of the many uses of the discrete Fourier transform (FT) in electronics communication systems is to analyse an arbitrary periodic time domain voltage waveform and convert it to the frequency domain [1]. Such a waveform may be mathematically decomposed into a sum of harmonically related sine waveforms, cosine waveforms and a constant (DC component), being known as a Fourier series. If a waveform is not periodic (also known as being aperiodic), such as a single pulse, we will see later that this may be approximated to a periodic waveform, but with a very long period.

## 1.1 An Example of an Arbitrary Periodic Waveform

Many voltage-time domain waveform measuring instruments, such as a storage oscilloscope, include a facility to save the data co-ordinates measured in digital form to an internal or external memory. In this case the co-ordinates would be voltage against time (V,t), with time being the independent variable and time the dependent variable. Such data are stored in the memory in discrete form, effectively a table time values and corresponding voltage values. Usually the number of data points has to be set up, with better precision being obtained for a larger number of points but at the expense of increased processing time.

Figure 1-1 shows a typical periodic voltage waveform as might be measured by a digital storage oscilloscope. To record the co-ordinates in this case, 101 data points were used covering a total (sample) time interval of one second and therefore a step size of 10 ms. The associated voltage time (V,t) co-ordinates for the waveform are shown in Table 1-1. Examination of these will show that this particular waveform has a period of 40 ms, so the voltage co-ordinates repeat every 40 ms. If the function describing the voltage variation with respect to time is $x(t)$, then another way of describing this periodicity is:

$$x(t) = x(t+T) = x(t+2T) = ... = x(t+kT) \qquad (1.1)$$

where

$x(t)$ is the periodic function with respect to time being considered;

$T = 40 \quad ms$ is the period of the waveform in this case;

$k$ is an integer.

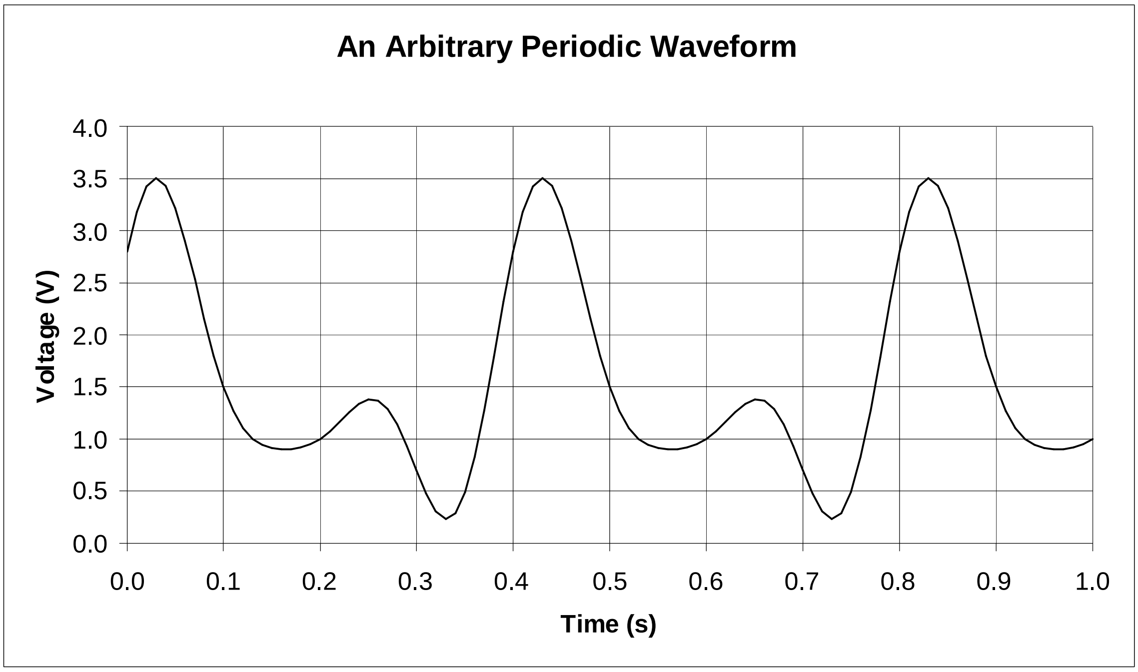


**Figure 1-1 An arbitrary, periodic, voltage against time waveform, as displayed on a digital storage oscilloscope, for analysis by discrete Fourier transform**

| t (s) | x(t) | t (s) | x(t) | t (s) | x(t) | t (s) | x(t) |
|---|---|---|---|---|---|---|---|
| 0.00 | 2.80 | 0.25 | 1.38 | 0.50 | 1.50 | 0.75 | 0.49 |
| 0.01 | 3.18 | 0.26 | 1.37 | 0.51 | 1.27 | 0.76 | 0.82 |
| 0.02 | 3.42 | 0.27 | 1.29 | 0.52 | 1.10 | 0.77 | 1.27 |
| 0.03 | 3.51 | 0.28 | 1.14 | 0.53 | 1.00 | 0.78 | 1.79 |
| 0.04 | 3.43 | 0.29 | 0.93 | 0.54 | 0.94 | 0.79 | 2.32 |
| 0.05 | 3.22 | 0.30 | 0.70 | 0.55 | 0.91 | 0.80 | 2.80 |
| 0.06 | 2.91 | 0.31 | 0.48 | 0.56 | 0.90 | 0.81 | 3.18 |
| 0.07 | 2.54 | 0.32 | 0.31 | 0.57 | 0.90 | 0.82 | 3.42 |
| 0.08 | 2.15 | 0.33 | 0.23 | 0.58 | 0.92 | 0.83 | 3.51 |
| 0.09 | 1.80 | 0.34 | 0.29 | 0.59 | 0.95 | 0.84 | 3.43 |
| 0.10 | 1.50 | 0.35 | 0.49 | 0.60 | 1.00 | 0.85 | 3.22 |
| 0.11 | 1.27 | 0.36 | 0.82 | 0.61 | 1.07 | 0.86 | 2.91 |
| 0.12 | 1.10 | 0.37 | 1.27 | 0.62 | 1.16 | 0.87 | 2.54 |
| 0.13 | 1.00 | 0.38 | 1.79 | 0.63 | 1.26 | 0.88 | 2.15 |
| 0.14 | 0.94 | 0.39 | 2.32 | 0.64 | 1.34 | 0.89 | 1.80 |
| 0.15 | 0.91 | 0.40 | 2.80 | 0.65 | 1.38 | 0.90 | 1.50 |
| 0.16 | 0.90 | 0.41 | 3.18 | 0.66 | 1.37 | 0.91 | 1.27 |
| 0.17 | 0.90 | 0.42 | 3.42 | 0.67 | 1.29 | 0.92 | 1.10 |
| 0.18 | 0.92 | 0.43 | 3.51 | 0.68 | 1.14 | 0.93 | 1.00 |
| 0.19 | 0.95 | 0.44 | 3.43 | 0.69 | 0.93 | 0.94 | 0.94 |
| 0.20 | 1.00 | 0.45 | 3.22 | 0.70 | 0.70 | 0.95 | 0.91 |
| 0.21 | 1.07 | 0.46 | 2.91 | 0.71 | 0.48 | 0.96 | 0.90 |
| 0.22 | 1.16 | 0.47 | 2.54 | 0.72 | 0.31 | 0.97 | 0.90 |
| 0.23 | 1.26 | 0.48 | 2.15 | 0.73 | 0.23 | 0.98 | 0.92 |
| 0.24 | 1.34 | 0.49 | 1.80 | 0.74 | 0.29 | 0.99 | 0.95 |
| | | | | | | 1.00 | 1.00 |

**Table 1-1 The voltage against time co-ordinates associated with the periodic waveform shown in Figure 1-1, as read from the memory of the digital storage oscilloscope.**

The Fourier series resulting from a Fourier transform can be expressed in the following way [7]:

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=0}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right) \\ &= A_0 + \sum_{n=1}^{\infty}\left[ A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right)\right] \end{aligned} \tag{1.2}$$

The first term of the series, for $n = 0$, $A_0$, is the DC, or zero frequency component. The other terms comprise a number of cosine terms and an equal and sine terms. In each case these are ordered as the fundamental, second harmonic, third harmonic and so on. Theoretically these continue up to the infinite harmonic but in practise usually only a few terms are necessary. There is also a $B_0$ term but this is always zero as it is a sine with a zero argument.

The arguments of the sine and cosine terms can be written differently, since the frequency $f$ of the original periodic voltage against time is given by

$$f = \frac{1}{T} \tag{1.3}$$

and the angular frequency of the original voltage time waveform $\omega_0$ is given by

$$\omega_0 = 2\pi f = \frac{2\pi}{T} \tag{1.4}$$

Using (1.4), (1.2) can be expanded out for the terms up to the third harmonic as

$$\begin{aligned} x(t) &= A_0 + \sum_{n=1}^{\infty}\left[ A_n \cos(n\omega t) + B_n \sin(n\omega t)\right] \\ &= A_0 + A_1 \cos(\omega_0 t) + A_2 \cos(2\omega_0 t) + A_3 \cos(3\omega_0 t) + \ldots \\ &\quad + B_1 \sin(\omega_0 t) + B_2 \sin(2\omega_0 t) + B_3 \sin(3\omega_0 t) + \ldots \end{aligned} \tag{1.5}$$

## 1.2 How do we calculate $A_0$ and the Fourier coefficients $A_1$, $A_2$, $B_1$, $B_2$ etc.?

From (1.5) the lower limit of the sum may be reduced to zero to embody the $A_0$ term within the summation, thus:

$$x(t) = \sum_{n=0}^{\infty}\left[ A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)\right] \tag{1.6}$$

A sometimes useful property of sinusoidal and co-sinusoidal waveforms is that the result of the integral with respect to time over the limits $t_1$ to $t_2$ where $t_2 - t_1 = T$ or one period, is always zero. This is exploited over the following few steps.

Multiply both sides of (1.6) by $\cos(p\omega_0 t)$ where $p$ is another positive integer and then take the integral of the results over the limits $t = 0$ to $t = T$.

$$\begin{aligned} \cos(p\omega_0 t)x(t) &= \cos(p\omega_0 t)\sum_{n=0}^{\infty}\left[ A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)\right] \\ \int_{t=0}^{T} \cos(p\omega_0 t)x(t)dt &= \int_{t=0}^{T} \cos(p\omega_0 t) \left[\sum_{n=0}^{\infty}\left[ A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)\right]\right]dt \end{aligned} \tag{1.7}$$

The right hand side of (1.7) can be re-arranged with the summation and integral symbols in different positions thus

$$\int_{t=0}^{T} \cos(p\omega_0 t) x(t) dt = \sum_{n=0}^{\infty} A_n \int_{t=0}^{T} \cos(p\omega_0 t)\cos(n\omega_0 t) dt + \sum_{n=0}^{\infty} B_n \int_{t=0}^{T} \cos(p\omega_0 t)\sin(n\omega_0 t) dt \tag{1.8}$$

Next, using the 'product of sine and cosine' trigonometry identity, that is

$$\sin P \cos Q = \frac{1}{2}\left[\sin(P+Q) + \sin(P-Q)\right] \tag{1.9}$$

the far right integral term of (1.8) evaluates to zero as follows:

$$\begin{aligned} & B_n \int_{t=0}^{T} \cos(p\omega_0 t)\sin(n\omega_0 t) dt \\ &= \frac{B_n}{2} \int_{t=0}^{T} \left[\sin\left[(n+p)\omega_0 t\right] + \sin\left[(n-p)\omega_0 t\right]\right] dt \\ &= \frac{B_n}{2}\left[-\frac{T}{(n+p)2\pi}\cos\left[\frac{(n+p)2\pi t}{T}\right]\right]_{t=0}^{T} + \frac{B_n}{2}\left[-\frac{T}{(n-p)2\pi}\cos\left[\frac{(n-p)2\pi t}{T}\right]\right]_{t=0}^{T} \\ &= \frac{-B_n T}{4\pi\left(n+p\right)}\left[\cos\left[(n+p)2\pi\right] - \cos 0\right] - \frac{B_n T}{4\pi\left(n-p\right)}\left[\cos\left[(n-p)2\pi\right] - \cos 0\right] \\ &= 0 \end{aligned} \tag{1.10}$$

Therefore (1.8) simplifies to

$$\int_{t=0}^{T} \cos(p\omega_0 t) x(t) dt = \sum_{n=0}^{\infty} A_n \int_{t=0}^{T} \cos(p\omega_0 t)\cos(n\omega_0 t) dt \tag{1.11}$$

To expand this we need to use the 'product of cosines' trigonometric identity, or

$$\cos P \cos Q = \frac{1}{2}\left[\cos(P+Q) + \cos(P-Q)\right] \tag{1.12}$$

The integral part of the right hand side of (1.11) evaluates to different results, dependent on the relationship between $p$ and $n$.

If $p \neq n$

$$\begin{aligned} A_n \int_{t=0}^{T} \cos(p\omega_0 t)\cos(n\omega_0 t) dt &= \frac{A_n}{2} \int_{t=0}^{T} \left[\cos\left[(p+n)\omega_0 t\right] + \cos\left[(p-n)\omega_0 t\right]\right] \\ &= 0 \end{aligned} \tag{1.13}$$

If $p = n$

$$
\begin{aligned}
A_n \int_{t=0}^{T} \cos(p\omega_0 t)\cos(n\omega_0 t)\,dt &= A_n \int_{t=0}^{T} \cos^2(n\omega_0 t)\,dt \\
&= \frac{A_n}{2}\int_{t=0}^{T} \left[1+\cos(2n\omega_0 t)\right]dt \\
&= \frac{A_n}{2}\int_{t=0}^{T} \left[1+\cos\left(\frac{4\pi nt}{T}\right)\right]dt \\
&= \frac{A_n}{2}\int_{t=0}^{T} dt + \frac{A_n}{2}\int_{t=0}^{T} \cos\left(\frac{4\pi nt}{T}\right)dt \qquad (1.14)\\
&= \frac{A_n T}{2} + \frac{A_n}{2}\left[\frac{T}{4\pi n}\sin\left(\frac{4\pi nt}{T}\right)\right]_{t=0}^{T} \\
&= \frac{A_n T}{2} + \frac{A_n T}{8\pi n}\left[\sin(4\pi n) - \sin(0)\right] \\
&= \frac{A_n T}{2}
\end{aligned}
$$

Now we can use the results of (1.10), (1.13) and (1.14) and substitute them into (1.8), which gives the result:

$$\int_{t=0}^{T} \cos(p\omega_0 t)\,x(t)\,dt = \frac{A_n T}{2} \qquad (1.15)$$

From (1.15) then, since $p = n$ in this case

$$A_n = \frac{2}{T}\int_{t=0}^{T} x(t)\cos(n\omega_0 t)\,dt \qquad (1.16)$$

By multiplying both sides of (1.8) by $\sin(p\omega_0 t)$, and performing some similar simplifications gives the following similar result:

$$B_n = \frac{2}{T}\int_{t=0}^{T} x(t)\sin(n\omega_0 t)\,dt \qquad (1.17)$$

Therefore, for a periodic waveform $x(t)$ of period $T$,

$$x(t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=0}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right) \qquad (1.18)$$

the cosine and sine coefficients are given respectively by

$$A_n = \frac{2}{T}\int_{t=0}^{T} x(t)\cos(n\omega_0 t)\,dt \qquad (1.19)$$

and

$$B_n = \frac{2}{T}\int_{t=0}^{T} x(t)\sin(n\omega_0 t)\,dt \qquad (1.20)$$

## 1.3 A Practical Example of the Fourier Decomposition of a Periodic Waveform

In Section 1.1 we saw in Figure 1-1 an example of an apparently periodic waveform of voltage against time. In order to decompose it into its Fourier components we have to first establish the necessary condition that it is in fact periodic. Here I must confess to having deliberately generated a periodic waveform for the purposes of this example with a period 400 ms ($T = 400\,ms$) and this indeed can be confirmed by examining the corresponding voltage and time co-ordinates which are shown in Table 1-1. Using an Excel spreadsheet, the voltage and time co-ordinates covering just one period are repeated in Table 1-3 together with other columns of intermediate values used to eventually determining the Fourier components themselves.

In Table 1-3, columns 1 to 4 contain the point reference $p$ starting at zero, the time in seconds, the fractional period and the value of $x$ at time $t$, $x(t)$ respectively. The two groups of columns headed COSINE and SINE contain values that are used to build up the cosine and sine coefficients for the first 4 terms (n = 0 to n = 3 in each case) of their respective series from (1.19) and (1.20). Although the Fourier series theoretically has an infinite number of terms, we shall see that as few as the four considered here is more than adequate for most purposes. In this example, the integration is being performed numerically and each row of the COSINE and SINE columns represents the expression to the right of the integral sign in (1.19) or (1.20) respectively. The time increment in each case (the integral step $dt$) is the same as the time interval from one point to the next, or 10 ms in this case. Of course the definition of the integral operator requires that $dt$ tends to zero but this approximation is quite acceptable for this illustration.

As a first example, consider the element in the second row ( $p = 1$) of the cosine column for the coefficient $A_0$. at $t = t_p = t_1$, represented by $\left(A_0\right)_{t=t_1}$. Then, using the expression to the right of the integral sign in (1.19),

$$\left(A_0\right)_{t=t_1} = x\left(t_1\right)\cos\left(n\omega_0 t_1\right)dt \tag{1.21}$$

Since we know that

$$\omega_0 = 2\pi f = \frac{2\pi}{T} \tag{1.22}$$

the cosine argument becomes $\frac{2\pi nt}{T}$. We are also using an integral time increment $dt$ equal to the time step, so

$$dt = t_p - t_{p-1} \tag{1.23}$$

(1.21) therefore becomes

$$\begin{aligned}\left(A_0\right)_{t=t_1} &= x\left(t_1\right)\cos\left(\frac{2\pi n t_1}{T}\right)\left(t_1 - t_0\right) \\ &= 3.182*\cos\left(\frac{2\pi*0*0.01}{0.4}\right)*\left(0.01-0\right) \\ &= 0.03182\end{aligned} \tag{1.24}$$

In this case, the Excel spreadsheet was set to display three decimal places which rounds off to $0.032$ , though internally calculations are performed to the full precision available.

In case choosing a cosine argument of zero is considered suspicious, we will look also at the $B_2$ sine coefficient for point 24 $p = 24$ . In this case the evaluation is shown below.

$$\begin{aligned}(B_2)_{t=t_{24}} &= x(t_{24})\sin\left(\frac{2\pi n t_{24}}{T}\right)(t_{24} - t_{23}) \\ &= 1.337 * \sin\left(\frac{2\pi * 2 * 0.24}{0.4}\right) * (0.240 - 0.230) \\ &= 0.0127\end{aligned} \qquad (1.25)$$

The rows at the bottom of the sine and cosine columns contain the final evaluations of the Fourier coefficients. Each is calculated from the sum of the values in the column above it, covering exactly one period (40 steps or 41 points), equivalent to the integral summation in (1.19) or (1.20) as appropriate. The summation is then multiplied by $2/T$ to give the final coefficients shown. For example, in the cosine (coefficient = $A_0$) column, the final result for $A_0$ is 1.50. This results from the summation of the values in the whole column from the value at point 0 (0.00) to that at point 40 (0.028) with the result multiplied by $2/T$ .

## 1.4 Reconstruction of the Decomposed Waveform

The trigonometric results in Table 1-1 are reproduced in Table 1-2. shows a summary of the first four cosine and sine coefficients which were calculated in Section 1.3. It may be re-assuring to reconstruct the original waveform from these using (1.18) to verify that we get back to something like Figure 1-1. In this case we are only using the first four coefficients (n = 0 to n = 3), so it will be interesting to see how closely the results of reconstruction match the original.

| A0 | A1 | A2 | A3 | | B0 | B1 | B2 | B3 |
|---|---|---|---|---|---|---|---|---|
| 1.50 | 0.70 | 0.40 | 0.20 | | 0 | 0.6 | 0.8 | 0.2 |

**Table 1-2 Coefficients up to n = 3 for the decomposed waveform**

Again an Excel spreadsheet serves as a useful tool to demonstrate this. The synthesis equations are reproduced below from (1.18)

$$\begin{aligned}x(t) &= \sum_{n=0}^{\infty}\left(A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right)\right) \\ &= A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right)\right) \\ &= A_0 + A_1\cos(\omega_0 t) + A_2\cos(2\omega_0 t) + A_3\cos(3\omega_0 t) + ... \\ &+ B_1\sin(\omega_0 t) + B_2\sin(2\omega_0 t) + B_3\sin(3\omega_0 t) + ...\end{aligned} \qquad (1.26)$$

Note that the arguments of the cosine and sine terms in (1.26) have no additional phase shifts. Table 1-4 shows a section of the Excel spreadsheet which was used to reconstruct the original waveform.

| | TIME | FRACT. PERIOD | VOLTAGE | FOURIER COEFFICIENTS | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | COSINE | | | | | SINE | | | | |
| | | | | | A0 | A1 | A2 | A3 | | B0 | B1 | B2 | B3 |
| Point (p) | t (s) | t/T | x(t) V | n = | 0 | 1 | 2 | 3 | | 0 | 1 | 2 | 3 |
| 0 | 0.000 | 0.000 | 2.800 | | | | | | | | | | |
| 1 | 0.010 | 0.025 | 3.182 | | 0.032 | 0.031 | 0.030 | 0.028 | | 0.000 | 0.005 | 0.010 | 0.014 |
| 2 | 0.020 | 0.050 | 3.424 | | 0.034 | 0.033 | 0.028 | 0.020 | | 0.000 | 0.011 | 0.020 | 0.028 |
| 3 | 0.030 | 0.075 | 3.507 | | 0.035 | 0.031 | 0.021 | 0.005 | | 0.000 | 0.016 | 0.028 | 0.035 |
| 4 | 0.040 | 0.100 | 3.432 | | 0.034 | 0.028 | 0.011 | -0.011 | | 0.000 | 0.020 | 0.033 | 0.033 |
| 5 | 0.050 | 0.125 | 3.219 | | 0.032 | 0.023 | 0.000 | -0.023 | | 0.000 | 0.023 | 0.032 | 0.023 |
| 6 | 0.060 | 0.150 | 2.906 | | 0.029 | 0.017 | -0.009 | -0.028 | | 0.000 | 0.024 | 0.028 | 0.009 |
| 7 | 0.070 | 0.175 | 2.536 | | 0.025 | 0.012 | -0.015 | -0.025 | | 0.000 | 0.023 | 0.021 | -0.004 |
| 8 | 0.080 | 0.200 | 2.154 | | 0.022 | 0.007 | -0.017 | -0.017 | | 0.000 | 0.020 | 0.013 | -0.013 |
| 9 | 0.090 | 0.225 | 1.800 | | 0.018 | 0.003 | -0.017 | -0.008 | | 0.000 | 0.018 | 0.006 | -0.016 |
| 10 | 0.100 | 0.250 | 1.500 | | 0.015 | 0.000 | -0.015 | 0.000 | | 0.000 | 0.015 | 0.000 | -0.015 |
| 11 | 0.110 | 0.275 | 1.268 | | 0.013 | -0.002 | -0.012 | 0.006 | | 0.000 | 0.013 | -0.004 | -0.011 |
| 12 | 0.120 | 0.300 | 1.105 | | 0.011 | -0.003 | -0.009 | 0.009 | | 0.000 | 0.011 | -0.006 | -0.006 |
| 13 | 0.130 | 0.325 | 1.001 | | 0.010 | -0.005 | -0.006 | 0.010 | | 0.000 | 0.009 | -0.008 | -0.002 |
| 14 | 0.140 | 0.350 | 0.942 | | 0.009 | -0.006 | -0.003 | 0.009 | | 0.000 | 0.008 | -0.009 | 0.003 |
| 15 | 0.150 | 0.375 | 0.912 | | 0.009 | -0.006 | 0.000 | 0.006 | | 0.000 | 0.006 | -0.009 | 0.006 |
| 16 | 0.160 | 0.400 | 0.901 | | 0.009 | -0.007 | 0.003 | 0.003 | | 0.000 | 0.005 | -0.009 | 0.009 |
| 17 | 0.170 | 0.425 | 0.903 | | 0.009 | -0.008 | 0.005 | -0.001 | | 0.000 | 0.004 | -0.007 | 0.009 |
| 18 | 0.180 | 0.450 | 0.917 | | 0.009 | -0.009 | 0.007 | -0.005 | | 0.000 | 0.003 | -0.005 | 0.007 |
| 19 | 0.190 | 0.475 | 0.948 | | 0.009 | -0.009 | 0.009 | -0.008 | | 0.000 | 0.001 | -0.003 | 0.004 |
| 20 | 0.200 | 0.500 | 1.000 | | 0.010 | -0.010 | 0.010 | -0.010 | | 0.000 | 0.000 | 0.000 | 0.000 |
| 21 | 0.210 | 0.525 | 1.073 | | 0.011 | -0.011 | 0.010 | -0.010 | | 0.000 | -0.002 | 0.003 | -0.005 |
| 22 | 0.220 | 0.550 | 1.163 | | 0.012 | -0.011 | 0.009 | -0.007 | | 0.000 | -0.004 | 0.007 | -0.009 |
| 23 | 0.230 | 0.575 | 1.257 | | 0.013 | -0.011 | 0.007 | -0.002 | | 0.000 | -0.006 | 0.010 | -0.012 |
| 24 | 0.240 | 0.600 | 1.337 | | 0.013 | -0.011 | 0.004 | 0.004 | | 0.000 | -0.008 | 0.013 | -0.013 |
| 25 | 0.250 | 0.625 | 1.381 | | 0.014 | -0.010 | 0.000 | 0.010 | | 0.000 | -0.010 | 0.014 | -0.010 |
| 26 | 0.260 | 0.650 | 1.369 | | 0.014 | -0.008 | -0.004 | 0.013 | | 0.000 | -0.011 | 0.013 | -0.004 |
| 27 | 0.270 | 0.675 | 1.289 | | 0.013 | -0.006 | -0.008 | 0.013 | | 0.000 | -0.011 | 0.010 | 0.002 |
| 28 | 0.280 | 0.700 | 1.139 | | 0.011 | -0.004 | -0.009 | 0.009 | | 0.000 | -0.011 | 0.007 | 0.007 |
| 29 | 0.290 | 0.725 | 0.934 | | 0.009 | -0.001 | -0.009 | 0.004 | | 0.000 | -0.009 | 0.003 | 0.008 |
| 30 | 0.300 | 0.750 | 0.700 | | 0.007 | 0.000 | -0.007 | 0.000 | | 0.000 | -0.007 | 0.000 | 0.007 |
| 31 | 0.310 | 0.775 | 0.477 | | 0.005 | 0.001 | -0.005 | -0.002 | | 0.000 | -0.005 | -0.001 | 0.004 |
| 32 | 0.320 | 0.800 | 0.308 | | 0.003 | 0.001 | -0.002 | -0.002 | | 0.000 | -0.003 | -0.002 | 0.002 |
| 33 | 0.330 | 0.825 | 0.235 | | 0.002 | 0.001 | -0.001 | -0.002 | | 0.000 | -0.002 | -0.002 | 0.000 |
| 34 | 0.340 | 0.850 | 0.290 | | 0.003 | 0.002 | -0.001 | -0.003 | | 0.000 | -0.002 | -0.003 | -0.001 |
| 35 | 0.350 | 0.875 | 0.488 | | 0.005 | 0.003 | 0.000 | -0.003 | | 0.000 | -0.003 | -0.005 | -0.003 |
| 36 | 0.360 | 0.900 | 0.824 | | 0.008 | 0.007 | 0.003 | -0.003 | | 0.000 | -0.005 | -0.008 | -0.008 |
| 37 | 0.370 | 0.925 | 1.273 | | 0.013 | 0.011 | 0.007 | 0.002 | | 0.000 | -0.006 | -0.010 | -0.013 |
| 38 | 0.380 | 0.950 | 1.789 | | 0.018 | 0.017 | 0.014 | 0.011 | | 0.000 | -0.006 | -0.011 | -0.014 |
| 39 | 0.390 | 0.975 | 2.318 | | 0.023 | 0.023 | 0.022 | 0.021 | | 0.000 | -0.004 | -0.007 | -0.011 |
| 40 | 0.400 | 1.000 | 2.800 | | 0.028 | 0.028 | 0.028 | 0.028 | | 0.000 | 0.000 | 0.000 | 0.000 |

**COSINE AND SINE SERIES REPRESENTATION**
**SUMS OF THE ABOVE COLUMNS (ZERO PHASE SHIFT)**

| A0 | A1 | A2 | A3 | | B0 | B1 | B2 | B3 |
|---|---|---|---|---|---|---|---|---|
| 1.50 | 0.70 | 0.40 | 0.20 | | 0 | 0.6 | 0.8 | 0.2 |

**MAGNITUDE/ANGLE REPRESENTATION**

| MAGNITUDE | | | | | ANGLE (rad) | | | |
|---|---|---|---|---|---|---|---|---|
| M0 | M1 | M2 | M3 | | Theta0 | Theta1 | Theta2 | Theta3 |
| 1.5 | 0.92195 | 0.89443 | 0.28284 | | 0 | 0.70863 | 1.10715 | 0.7854 |

**Table 1-3 A tabular illustration of how the Fourier coefficients up to $n=3$ may be calculated for an arbitrary periodic voltage time waveform of period 400 ms that is also shown in Figure 1-1. The results are shown for both the trigonometric (cosine and sine) format and the polar (magnitude and angle) format.**

**Periodic waveform synthesis**

| t (s) | A0 | A1... | A2... | A3... | B0 | B1... | B2... | B3... | x(t) |
|---|---|---|---|---|---|---|---|---|---|
| 0.00 | 1.50 | 0.70 | 0.40 | 0.20 | 0.00 | 0.00 | 0.00 | 0.00 | 2.80 |
| 0.01 | 1.50 | 0.69 | 0.38 | 0.18 | 0.00 | 0.09 | 0.25 | 0.09 | 3.18 |
| 0.02 | 1.50 | 0.67 | 0.32 | 0.12 | 0.00 | 0.19 | 0.47 | 0.16 | 3.42 |
| 0.03 | 1.50 | 0.62 | 0.24 | 0.03 | 0.00 | 0.27 | 0.65 | 0.20 | 3.51 |
| 0.04 | 1.50 | 0.57 | 0.12 | -0.06 | 0.00 | 0.35 | 0.76 | 0.19 | 3.43 |
| 0.05 | 1.50 | 0.49 | 0.00 | -0.14 | 0.00 | 0.42 | 0.80 | 0.14 | 3.22 |
| 0.06 | 1.50 | 0.41 | -0.12 | -0.19 | 0.00 | 0.49 | 0.76 | 0.06 | 2.91 |
| 0.07 | 1.50 | 0.32 | -0.24 | -0.20 | 0.00 | 0.53 | 0.65 | -0.03 | 2.54 |
| 0.08 | 1.50 | 0.22 | -0.32 | -0.16 | 0.00 | 0.57 | 0.47 | -0.12 | 2.15 |
| 0.09 | 1.50 | 0.11 | -0.38 | -0.09 | 0.00 | 0.59 | 0.25 | -0.18 | 1.80 |
| 0.10 | 1.50 | 0.00 | -0.40 | 0.00 | 0.00 | 0.60 | 0.00 | -0.20 | 1.50 |
| 0.11 | 1.50 | -0.11 | -0.38 | 0.09 | 0.00 | 0.59 | -0.25 | -0.18 | 1.27 |
| 0.12 | 1.50 | -0.22 | -0.32 | 0.16 | 0.00 | 0.57 | -0.47 | -0.12 | 1.10 |
| 0.13 | 1.50 | -0.32 | -0.24 | 0.20 | 0.00 | 0.53 | -0.65 | -0.03 | 1.00 |
| 0.14 | 1.50 | -0.41 | -0.12 | 0.19 | 0.00 | 0.49 | -0.76 | 0.06 | 0.94 |
| 0.15 | 1.50 | -0.49 | 0.00 | 0.14 | 0.00 | 0.42 | -0.80 | 0.14 | 0.91 |
| 0.16 | 1.50 | -0.57 | 0.12 | 0.06 | 0.00 | 0.35 | -0.76 | 0.19 | 0.90 |
| 0.17 | 1.50 | -0.62 | 0.24 | -0.03 | 0.00 | 0.27 | -0.65 | 0.20 | 0.90 |
| 0.18 | 1.50 | -0.67 | 0.32 | -0.12 | 0.00 | 0.19 | -0.47 | 0.16 | 0.92 |
| 0.19 | 1.50 | -0.69 | 0.38 | -0.18 | 0.00 | 0.09 | -0.25 | 0.09 | 0.95 |
| 0.20 | 1.50 | -0.70 | 0.40 | -0.20 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 |
| 0.21 | 1.50 | -0.69 | 0.38 | -0.18 | 0.00 | -0.09 | 0.25 | -0.09 | 1.07 |
| 0.22 | 1.50 | -0.67 | 0.32 | -0.12 | 0.00 | -0.19 | 0.47 | -0.16 | 1.16 |
| 0.23 | 1.50 | -0.62 | 0.24 | -0.03 | 0.00 | -0.27 | 0.65 | -0.20 | 1.26 |
| 0.24 | 1.50 | -0.57 | 0.12 | 0.06 | 0.00 | -0.35 | 0.76 | -0.19 | 1.34 |
| 0.25 | 1.50 | -0.49 | 0.00 | 0.14 | 0.00 | -0.42 | 0.80 | -0.14 | 1.38 |
| 0.26 | 1.50 | -0.41 | -0.12 | 0.19 | 0.00 | -0.49 | 0.76 | -0.06 | 1.37 |
| 0.27 | 1.50 | -0.32 | -0.24 | 0.20 | 0.00 | -0.53 | 0.65 | 0.03 | 1.29 |
| 0.28 | 1.50 | -0.22 | -0.32 | 0.16 | 0.00 | -0.57 | 0.47 | 0.12 | 1.14 |
| 0.29 | 1.50 | -0.11 | -0.38 | 0.09 | 0.00 | -0.59 | 0.25 | 0.18 | 0.93 |
| 0.30 | 1.50 | 0.00 | -0.40 | 0.00 | 0.00 | -0.60 | 0.00 | 0.20 | 0.70 |
| 0.31 | 1.50 | 0.11 | -0.38 | -0.09 | 0.00 | -0.59 | -0.25 | 0.18 | 0.48 |
| 0.32 | 1.50 | 0.22 | -0.32 | -0.16 | 0.00 | -0.57 | -0.47 | 0.12 | 0.31 |
| 0.33 | 1.50 | 0.32 | -0.24 | -0.20 | 0.00 | -0.53 | -0.65 | 0.03 | 0.23 |
| 0.34 | 1.50 | 0.41 | -0.12 | -0.19 | 0.00 | -0.49 | -0.76 | -0.06 | 0.29 |
| 0.35 | 1.50 | 0.49 | 0.00 | -0.14 | 0.00 | -0.42 | -0.80 | -0.14 | 0.49 |
| 0.36 | 1.50 | 0.57 | 0.12 | -0.06 | 0.00 | -0.35 | -0.76 | -0.19 | 0.82 |
| 0.37 | 1.50 | 0.62 | 0.24 | 0.03 | 0.00 | -0.27 | -0.65 | -0.20 | 1.27 |
| 0.38 | 1.50 | 0.67 | 0.32 | 0.12 | 0.00 | -0.19 | -0.47 | -0.16 | 1.79 |
| 0.39 | 1.50 | 0.69 | 0.38 | 0.18 | 0.00 | -0.09 | -0.25 | -0.09 | 2.32 |
| 0.40 | 1.50 | 0.70 | 0.40 | 0.20 | 0.00 | 0.00 | 0.00 | 0.00 | 2.80 |
| 0.41 | 1.50 | 0.69 | 0.38 | 0.18 | 0.00 | 0.09 | 0.25 | 0.09 | 3.18 |
| 0.42 | 1.50 | 0.67 | 0.32 | 0.12 | 0.00 | 0.19 | 0.47 | 0.16 | 3.42 |
| 0.43 | 1.50 | 0.62 | 0.24 | 0.03 | 0.00 | 0.27 | 0.65 | 0.20 | 3.51 |
| 0.44 | 1.50 | 0.57 | 0.12 | -0.06 | 0.00 | 0.35 | 0.76 | 0.19 | 3.43 |
| 0.45 | 1.50 | 0.49 | 0.00 | -0.14 | 0.00 | 0.42 | 0.80 | 0.14 | 3.22 |
| 0.46 | 1.50 | 0.41 | -0.12 | -0.19 | 0.00 | 0.49 | 0.76 | 0.06 | 2.91 |
| 0.47 | 1.50 | 0.32 | -0.24 | -0.20 | 0.00 | 0.53 | 0.65 | -0.03 | 2.54 |
| 0.48 | 1.50 | 0.22 | -0.32 | -0.16 | 0.00 | 0.57 | 0.47 | -0.12 | 2.15 |
| 0.49 | 1.50 | 0.11 | -0.38 | -0.09 | 0.00 | 0.59 | 0.25 | -0.18 | 1.80 |
| 0.50 | 1.50 | 0.00 | -0.40 | 0.00 | 0.00 | 0.60 | 0.00 | -0.20 | 1.50 |
| 0.51 | 1.50 | -0.11 | -0.38 | 0.09 | 0.00 | 0.59 | -0.25 | -0.18 | 1.27 |
| 0.52 | 1.50 | -0.22 | -0.32 | 0.16 | 0.00 | 0.57 | -0.47 | -0.12 | 1.10 |
| 0.53 | 1.50 | -0.32 | -0.24 | 0.20 | 0.00 | 0.53 | -0.65 | -0.03 | 1.00 |
| 0.54 | 1.50 | -0.41 | -0.12 | 0.19 | 0.00 | 0.49 | -0.76 | 0.06 | 0.94 |
| 0.55 | 1.50 | -0.49 | 0.00 | 0.14 | 0.00 | 0.42 | -0.80 | 0.14 | 0.91 |
| 0.56 | 1.50 | -0.57 | 0.12 | 0.06 | 0.00 | 0.35 | -0.76 | 0.19 | 0.90 |
| 0.57 | 1.50 | -0.62 | 0.24 | -0.03 | 0.00 | 0.27 | -0.65 | 0.20 | 0.90 |
| 0.58 | 1.50 | -0.67 | 0.32 | -0.12 | 0.00 | 0.19 | -0.47 | 0.16 | 0.92 |
| 0.59 | 1.50 | -0.69 | 0.38 | -0.18 | 0.00 | 0.09 | -0.25 | 0.09 | 0.95 |
| 0.60 | 1.50 | -0.70 | 0.40 | -0.20 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 |
| 0.61 | 1.50 | -0.69 | 0.38 | -0.18 | 0.00 | -0.09 | 0.25 | -0.09 | 1.07 |
| 0.62 | 1.50 | -0.67 | 0.32 | -0.12 | 0.00 | -0.19 | 0.47 | -0.16 | 1.16 |
| 0.63 | 1.50 | -0.62 | 0.24 | -0.03 | 0.00 | -0.27 | 0.65 | -0.20 | 1.26 |
| 0.64 | 1.50 | -0.57 | 0.12 | 0.06 | 0.00 | -0.35 | 0.76 | -0.19 | 1.34 |
| 0.65 | 1.50 | -0.49 | 0.00 | 0.14 | 0.00 | -0.42 | 0.80 | -0.14 | 1.38 |
| 0.66 | 1.50 | -0.41 | -0.12 | 0.19 | 0.00 | -0.49 | 0.76 | -0.06 | 1.37 |
| 0.67 | 1.50 | -0.32 | -0.24 | 0.20 | 0.00 | -0.53 | 0.65 | 0.03 | 1.29 |
| 0.68 | 1.50 | -0.22 | -0.32 | 0.16 | 0.00 | -0.57 | 0.47 | 0.12 | 1.14 |
| 0.69 | 1.50 | -0.11 | -0.38 | 0.09 | 0.00 | -0.59 | 0.25 | 0.18 | 0.93 |

**Periodic waveform synthesis**

| t (s) | A0 | A1... | A2... | A3... | B0 | B1... | B2... | B3... | x(t) |
|---|---|---|---|---|---|---|---|---|---|
| 0.70 | 1.50 | 0.00 | -0.40 | 0.00 | 0.00 | -0.60 | 0.00 | 0.20 | 0.70 |
| 0.71 | 1.50 | 0.11 | -0.38 | -0.09 | 0.00 | -0.59 | -0.25 | 0.18 | 0.48 |
| 0.72 | 1.50 | 0.22 | -0.32 | -0.16 | 0.00 | -0.57 | -0.47 | 0.12 | 0.31 |
| 0.73 | 1.50 | 0.32 | -0.24 | -0.20 | 0.00 | -0.53 | -0.65 | 0.03 | 0.23 |
| 0.74 | 1.50 | 0.41 | -0.12 | -0.19 | 0.00 | -0.49 | -0.76 | -0.06 | 0.29 |
| 0.75 | 1.50 | 0.49 | 0.00 | -0.14 | 0.00 | -0.42 | -0.80 | -0.14 | 0.49 |
| 0.76 | 1.50 | 0.57 | 0.12 | -0.06 | 0.00 | -0.35 | -0.76 | -0.19 | 0.82 |
| 0.77 | 1.50 | 0.62 | 0.24 | 0.03 | 0.00 | -0.27 | -0.65 | -0.20 | 1.27 |
| 0.78 | 1.50 | 0.67 | 0.32 | 0.12 | 0.00 | -0.19 | -0.47 | -0.16 | 1.79 |
| 0.79 | 1.50 | 0.69 | 0.38 | 0.18 | 0.00 | -0.09 | -0.25 | -0.09 | 2.32 |
| 0.80 | 1.50 | 0.70 | 0.40 | 0.20 | 0.00 | 0.00 | 0.00 | 0.00 | 2.80 |
| 0.81 | 1.50 | 0.69 | 0.38 | 0.18 | 0.00 | 0.09 | 0.25 | 0.09 | 3.18 |
| 0.82 | 1.50 | 0.67 | 0.32 | 0.12 | 0.00 | 0.19 | 0.47 | 0.16 | 3.42 |
| 0.83 | 1.50 | 0.62 | 0.24 | 0.03 | 0.00 | 0.27 | 0.65 | 0.20 | 3.51 |
| 0.84 | 1.50 | 0.57 | 0.12 | -0.06 | 0.00 | 0.35 | 0.76 | 0.19 | 3.43 |
| 0.85 | 1.50 | 0.49 | 0.00 | -0.14 | 0.00 | 0.42 | 0.80 | 0.14 | 3.22 |
| 0.86 | 1.50 | 0.41 | -0.12 | -0.19 | 0.00 | 0.49 | 0.76 | 0.06 | 2.91 |
| 0.87 | 1.50 | 0.32 | -0.24 | -0.20 | 0.00 | 0.53 | 0.65 | -0.03 | 2.54 |
| 0.88 | 1.50 | 0.22 | -0.32 | -0.16 | 0.00 | 0.57 | 0.47 | -0.12 | 2.15 |
| 0.89 | 1.50 | 0.11 | -0.38 | -0.09 | 0.00 | 0.59 | 0.25 | -0.18 | 1.80 |
| 0.90 | 1.50 | 0.00 | -0.40 | 0.00 | 0.00 | 0.60 | 0.00 | -0.20 | 1.50 |
| 0.91 | 1.50 | -0.11 | -0.38 | 0.09 | 0.00 | 0.59 | -0.25 | -0.18 | 1.27 |
| 0.92 | 1.50 | -0.22 | -0.32 | 0.16 | 0.00 | 0.57 | -0.47 | -0.12 | 1.10 |
| 0.93 | 1.50 | -0.32 | -0.24 | 0.20 | 0.00 | 0.53 | -0.65 | -0.03 | 1.00 |
| 0.94 | 1.50 | -0.41 | -0.12 | 0.19 | 0.00 | 0.49 | -0.76 | 0.06 | 0.94 |
| 0.95 | 1.50 | -0.49 | 0.00 | 0.14 | 0.00 | 0.42 | -0.80 | 0.14 | 0.91 |
| 0.96 | 1.50 | -0.57 | 0.12 | 0.06 | 0.00 | 0.35 | -0.76 | 0.19 | 0.90 |
| 0.97 | 1.50 | -0.62 | 0.24 | -0.03 | 0.00 | 0.27 | -0.65 | 0.20 | 0.90 |
| 0.98 | 1.50 | -0.67 | 0.32 | -0.12 | 0.00 | 0.19 | -0.47 | 0.16 | 0.92 |
| 0.99 | 1.50 | -0.69 | 0.38 | -0.18 | 0.00 | 0.09 | -0.25 | 0.09 | 0.95 |
| 1.00 | 1.50 | -0.70 | 0.40 | -0.20 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 |

**Table 1-4 Synthesis of the original waveform using the Fourier coefficients using an Excel spreadsheet**

The resulting cosine waveforms are shown in Figure 1-2 and sine waveforms are shown in Figure 1-3.

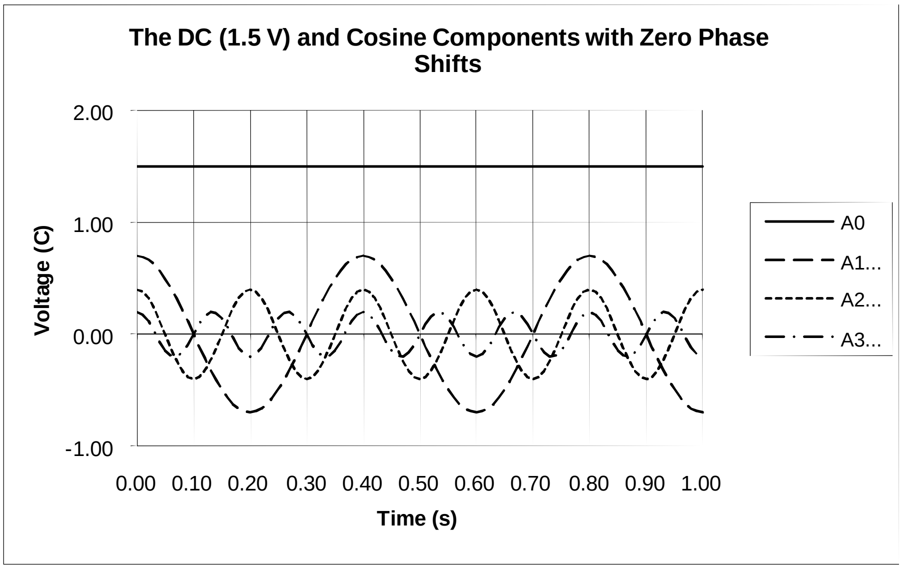


**Figure 1-2 Graphs of the cosine waveforms used to reconstruct the original waveform**

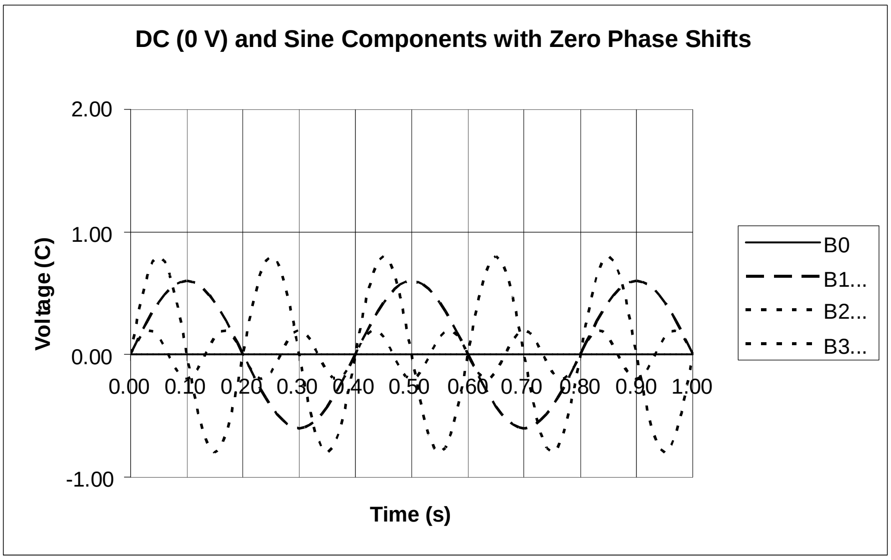


**Figure 1-3 Graphs of the sine waveforms used to reconstruct the original waveform**

The result of adding the constituent cosine and sine waveforms, point by point, in Figure 1-2 and Figure 1-3 is shown in Figure 1-4. Figure 1-4 is a close resemblance to the waveform that was originally analysed, shown in Figure 1-1.

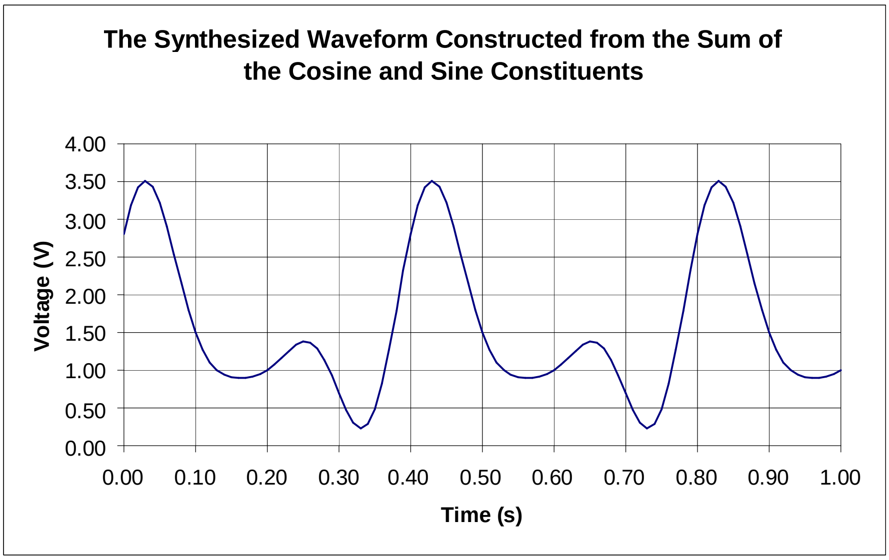


**Figure 1-4 The waveform constructed from the decomposed components**

## 1.5 Representation of the Fourier Series in Cosine Only or Sine Only Terms

In Section 1.3, the periodic waveform was described by a series comprising cosine terms and sine terms as shown in Figure 1-2 and Figure 1-3. These were pure cosine and sine wave respectively, neither had any phase offset (lead or lag) term in the argument. A pure cosine wave starts at unity for zero phase and a pure sinewave starts at zero for zero phase.

Sometimes it is useful to represent a Fourier series by just either cosine or sine terms. In this case, each term would require a unique coefficient (amplitude) and a unique argument (phase) term.

Returning to the general Fourier series, this is made up from pairs of cosine terms and pairs of sine terms, each pair having arguments representing the same frequency, progressively increasing harmonics. The associated sine and cosine amplitudes (coefficients) may not necessarily be equal but they can be represented as a cosine (or sine) expression with a modified amplitude and argument. The following reasoning applied to one pair of such terms with the same frequency shows how this may be achieved:

$$
\begin{aligned}
A\cos\omega t + B\sin\omega t &= \frac{\sqrt{A^2+B^2}}{\sqrt{A^2+B^2}}\left[A\cos\omega t + B\sin\omega t\right] \\
&= M\left[\frac{A}{\sqrt{A^2+B^2}}\cos\omega t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega t\right] \\
&= M\left[\cos\theta\cos\omega t + \sin\theta\sin\omega t\right] \qquad (1.27)\\
&= M\cos\left(\omega t - \theta\right) \\
&= \sqrt{A^2+B^2}\cos\left[\omega t - \tan^{-1}\frac{B}{A}\right]
\end{aligned}
$$

(1.26) may be written in the following generic way:

$$
\begin{aligned}
x(t) &= \sum_{n=0}^{\infty}\left(A_n\cos\left(\frac{2\pi nt}{T}\right) + B_n\sin\left(\frac{2\pi nt}{T}\right)\right) \\
&= \sum_{n=0}^{\infty} M_n\cos\left(n\omega_0 t - \theta_n\right)
\end{aligned}
\qquad (1.28)
$$

Using (1.27) and (1.28) show that the original equation $x(t)$ may be expressed in the alternative form shown, a sum of cosine terms whose amplitudes and phase shift components are given in terms of $n$ as follows:

$$\theta_n = \tan^{-1}\left(\frac{B_n}{A_n}\right) \qquad (1.29)$$

and

$$M_n = \sqrt{A_n^{\,2} + B_n^{\,2}} \qquad (1.30)$$

**Table 1-5** shows the results of calculating $M_n$ and $\theta_n$ for $n=0$ to $n=3$.

| COSINE AND SINE SERIES REPRESENTATION<br>SUMS OF THE ABOVE COLUMNS (ZERO PHASE SHIFT) | | | | | | | |
|---|---|---|---|---|---|---|---|
| A0 | A1 | A2 | A3 | B0 | B1 | B2 | B3 |
| 1.50 | 0.70 | 0.40 | 0.20 | 0 | 0.6 | 0.8 | 0.2 |

| MAGNITUDE/ANGLE REPRESENTATION | | | | | | | |
|---|---|---|---|---|---|---|---|
| MAGNITUDE | | | | ANGLE (rad) | | | |
| M0 | M1 | M2 | M3 | Theta0 | Theta1 | Theta2 | Theta3 |
| 1.500 | 0.922 | 0.894 | 0.283 | 0.000 | 0.709 | 1.107 | 0.785 |

**Table 1-5 Summary of the first four cosine and sine coefficients and the corresponding magnitude/angle results calculated from them using (1.29) and (1.30)**

**.**

| t (s) | x(t) | t (s) | x(t) | t (s) | x(t) | t (s) | x(t) |
|---|---|---|---|---|---|---|---|
| 0.00 | 2.800 | 0.25 | 1.381 | 0.50 | 1.500 | 0.75 | 0.488 |
| 0.01 | 3.182 | 0.26 | 1.369 | 0.51 | 1.268 | 0.76 | 0.824 |
| 0.02 | 3.424 | 0.27 | 1.289 | 0.52 | 1.105 | 0.77 | 1.273 |
| 0.03 | 3.507 | 0.28 | 1.139 | 0.53 | 1.001 | 0.78 | 1.789 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.04 | 3.432 | 0.29 | 0.934 | 0.54 | 0.942 | 0.79 | 2.318 |
| 0.05 | 3.219 | 0.30 | 0.700 | 0.55 | 0.912 | 0.80 | 2.800 |
| 0.06 | 2.906 | 0.31 | 0.477 | 0.56 | 0.901 | 0.81 | 3.182 |
| 0.07 | 2.536 | 0.32 | 0.308 | 0.57 | 0.903 | 0.82 | 3.424 |
| 0.08 | 2.154 | 0.33 | 0.235 | 0.58 | 0.917 | 0.83 | 3.507 |
| 0.09 | 1.800 | 0.34 | 0.290 | 0.59 | 0.948 | 0.84 | 3.432 |
| 0.10 | 1.500 | 0.35 | 0.488 | 0.60 | 1.000 | 0.85 | 3.219 |
| 0.11 | 1.268 | 0.36 | 0.824 | 0.61 | 1.073 | 0.86 | 2.906 |
| 0.12 | 1.105 | 0.37 | 1.273 | 0.62 | 1.163 | 0.87 | 2.536 |
| 0.13 | 1.001 | 0.38 | 1.789 | 0.63 | 1.257 | 0.88 | 2.154 |
| 0.14 | 0.942 | 0.39 | 2.318 | 0.64 | 1.337 | 0.89 | 1.800 |
| 0.15 | 0.912 | 0.40 | 2.800 | 0.65 | 1.381 | 0.90 | 1.500 |
| 0.16 | 0.901 | 0.41 | 3.182 | 0.66 | 1.369 | 0.91 | 1.268 |
| 0.17 | 0.903 | 0.42 | 3.424 | 0.67 | 1.289 | 0.92 | 1.105 |
| 0.18 | 0.917 | 0.43 | 3.507 | 0.68 | 1.139 | 0.93 | 1.001 |
| 0.19 | 0.948 | 0.44 | 3.432 | 0.69 | 0.934 | 0.94 | 0.942 |
| 0.20 | 1.000 | 0.45 | 3.219 | 0.70 | 0.700 | 0.95 | 0.912 |
| 0.21 | 1.073 | 0.46 | 2.906 | 0.71 | 0.477 | 0.96 | 0.901 |
| 0.22 | 1.163 | 0.47 | 2.536 | 0.72 | 0.308 | 0.97 | 0.903 |
| 0.23 | 1.257 | 0.48 | 2.154 | 0.73 | 0.235 | 0.98 | 0.917 |
| 0.24 | 1.337 | 0.49 | 1.800 | 0.74 | 0.290 | 0.99 | 0.948 |
| | | | | | | 1.00 | 1.000 |

**Table 1-6 A table of the results of substitution of the $M_n$ and $\theta_n$ values from Table 1-5 into (1.28)**

Table 1-6 shows values obtained for $x(t)$ using 10 ms steps over a duration or 1 s using the results from **Table 1-5**. It closely matches the original waveform in Figure 1-1.
Equation Section (Next)

## 2 THE DISCRETE COMPLEX FOURIER TRANSFORM

### 2.1 Eulers Equations

Euler's equations provide a conversion between complex numbers which are expressed in rectangular format and those expressed in exponential format. (2.1) shows the conversion from rectangular to exponential and (2.2) the conversion from exponential to rectangular.

$$\begin{aligned} e^{jx} &= \cos x + j\sin x \\ e^{-jx} &= \cos x - j\sin x \end{aligned} \qquad (2.1)$$

$$\begin{aligned} \cos x &= \frac{e^{jx} + e^{-jx}}{2} \\ \sin x &= \frac{e^{jx} - e^{-jx}}{2j} \end{aligned} \qquad (2.2)$$

In electrical engineering a convention has been adopted for representing co-sinusoidal waveforms in complex exponential format. For example, a time-varying co-sinusoidal waveform of instantaneous value $V_{inst}$, amplitude $V_0$, angular frequency $\omega$ could be written as

$$V_{inst} = V_0 \cos \omega t \qquad (2.3)$$

Using (2.1), this may be written in exponential format as

$$V_{inst} = V_0 \operatorname{Re}(e^{j\omega t}) \qquad (2.4)$$

where $\operatorname{Re}(e^{j\omega t})$ means 'the real part of $e^{j\omega t}$'. In fact the real part operator is normally omitted and an expression such that in (2.4) is understood from simply

$$V_{inst} = V_0 e^{j\omega t} \qquad (2.5)$$

If the waveform was sinusoidal the exponential expression would therefore be

$$V_{inst} = jV_0 e^{-j\omega t} \qquad (2.6)$$

Dealing with complex quantities in exponential as opposed to rectangular format allows us to use a briefer notation and makes mathematical manipulations such as calculus less onerous. Applying the rules of complex algebra and calculus correctly will yield a result that is also a complex number. In most cases, interpretation of the result in the real world requires us to consider the real part only, ignoring the imaginary part.

### 2.2 Fourier's Series in Complex Exponential Notation

By using the cosine and sine expressions in (2.2), the real Fourier series (1.28) may be transformed to the equivalent complex exponential form below:

$$\begin{aligned} x(t) &= A_0 + \sum_{n=1}^{\infty}\left[A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)\right] \\ &= A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n - jB_n}{2}\right)e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\left(\frac{A_n + jB_n}{2}\right)e^{-jn\omega_0 t} \end{aligned} \qquad (2.7)$$

In the second summation, we may move the negative sign from the exponential power to the summation limits, which therefore change to $-\infty$ and -1. The result is

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n - jB_n}{2}\right)e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1}\left(\frac{A_n + jB_n}{2}\right)e^{jn\omega_0 t} \tag{2.8}$$

Therefore, in general

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \tag{2.9}$$

and

$$C_0 = A_0 \tag{2.10}$$

For $n \succ 0$

$$C_n = \frac{A_n - jB_n}{2} \tag{2.11}$$

For $n \prec 0$

$$C_n = \frac{A_n + jB_n}{2} \tag{2.12}$$

For $n \geq 0$ , using (2.11) and the previously derived results for $A_n$ and $B_n$, (1.16) and (1.17) respectively, then

$$\begin{aligned} C_n &= \frac{A_n - jB_n}{2} \\ &= \frac{1}{T}\int_{t=0}^{T} x(t)\cos(n\omega_0 t)dt - \frac{j}{2}\int_{t=0}^{T} x(t)\sin(n\omega_0 t)dt \\ &= \frac{1}{T}\int_{t=0}^{T} x(t)[\cos(n\omega_0 t) - j\sin(n\omega_0 t)]dt \\ &= \frac{1}{T}\int_{t=0}^{T} x(t)e^{-jn\omega_0 t}dt \end{aligned} \tag{2.13}$$

When $n \prec 0$ it can be seen from (2.11) and (2.12) that the result for $C_n$ is the complex conjugate of the same result for $n \geq 0$ , therefore

$$C_n = C_{-n}^{\ *} \tag{2.14}$$

where the superscript asterisk indicates a complex conjugate.

When a periodic waveform is decomposed into its *complex* Fourier series, it will comprise a set of real parts and a set of imaginary parts. Each will have positive and negative frequency components. The real components will all have positive amplitudes but the complex components will have either positive amplitudes for positive frequencies or negative amplitudes for negative frequencies. The amplitudes for all $n = 1$ terms will be equal as will those for all $n = 2$ terms and so on. Figure 2-1 shows both cases for the first 3 non-zero terms in a Fourier series.

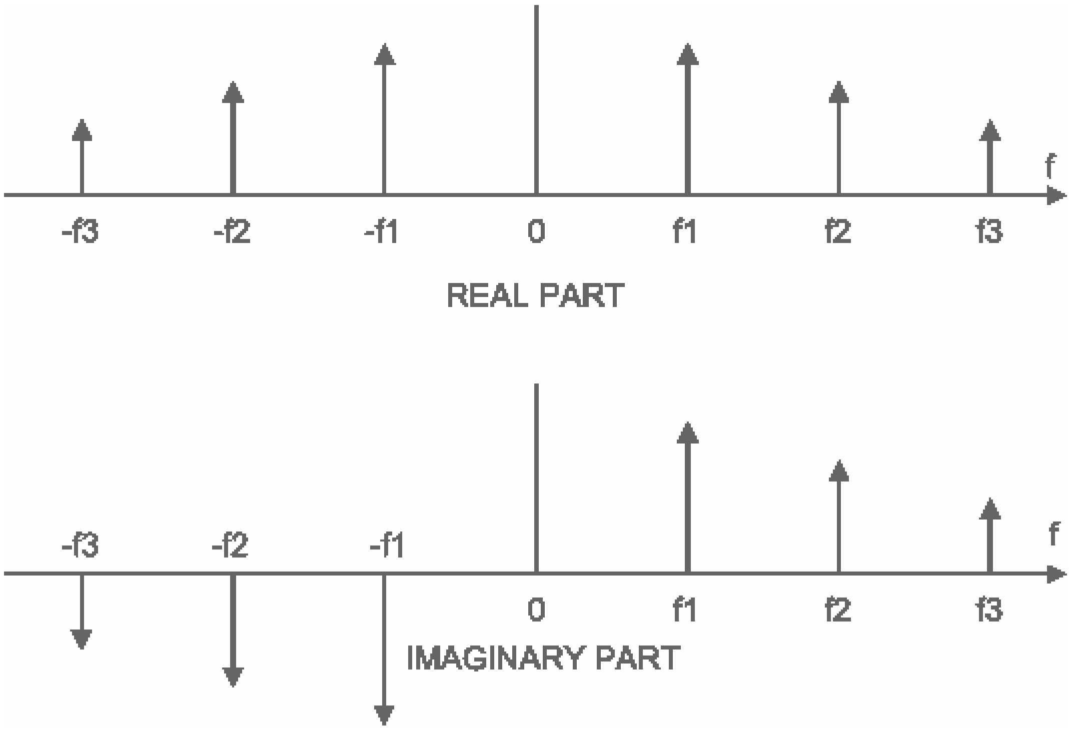


**Figure 2-1 Frequency amplitude planes for the real and imaginary parts of a complex Fourier series, covering the first three terms**

From Figure 2-1 we can see that, by adding the real and complex parts, the negative frequency components cancel out since each term has a positive (real) part and a negative (imaginary) part both of which are of equal amplitude. Similarly, the positive frequency terms add to give products, each of which is twice its original amplitude.

As with the real Fourier series, the magnitude $M_n$ and phase in radians $\theta_n$ are given by the following expressions:

$$M_n = \sqrt{A_n^{\ 2} + B_n^{\ 2}} \qquad (2.15)$$

and

$$\theta_n = \tan^{-1} \frac{B_n}{A_n} \qquad (2.16)$$

## 2.3 The Complex Fourier Transform of a Voltage Against Time Square Wave

Figure 2-2 shows a voltage against time square wave. This is the particular case of a pulse waveform with a peak voltage amplitude of 1 V and duty ratio of 50%. We have re-named the voltage axis $V(t)$ instead of $x(t)$ simply because it is more descriptive for a voltage.

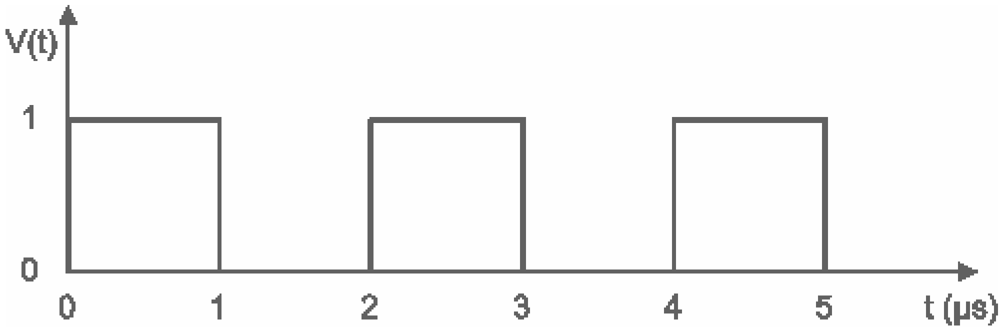


**Figure 2-2 Square Wave (50% Duty Cycle) of Voltage Against Time**

From (2.13), substituting $n=0$ for the constant (or DC) component:

$$C_n = \frac{1}{T}\int_{t=0}^{T} x(t)e^{-jn\omega_0 t} = \frac{1}{T}\int_{t=0}^{T} x(t)e^{-0}dt = \frac{1}{T}\int_{t=0}^{T/2} dt = \frac{1}{T}\left[t\right]_0^{T/2} = \frac{1}{2} \qquad (2.17)$$

For $n \neq 0$:

$$\begin{aligned} C_n &= \frac{1}{T}\int_{t=0}^{T} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_{t=0}^{T} e^{j\frac{2\pi nt}{T}}dt + 0 \\ &= \frac{1}{T}\left[\frac{-T}{j2\pi n}e^{\frac{-j2\pi nt}{T}}\right]_0^{T/2} = \frac{-1}{j2\pi n}\left[e^{\frac{-j2\pi nT}{2T}} - 1\right] \\ &= \frac{1}{-j2\pi n}\left[e^{-j\pi n} - 1\right] = \left(\frac{e^{\frac{j\pi n}{2}} - e^{\frac{-j\pi n}{2}}}{2j\pi n}\right)e^{\frac{-j\pi n}{2}} \\ &= \frac{\sin\left(\frac{\pi n}{2}\right)}{n\pi}e^{\frac{-j\pi n}{2}} \end{aligned} \qquad (2.18)$$

To obtain the voltage magnitude representation in the frequency plane (as opposed to the time plane that was shown in Figure 2-2) it may be easier to convert the result of (2.18) into real and imaginary sinusoidal and co-sinusoidal (or trigonometric) terms using the following forms of Euler's equations (2.1):

$$e^{\frac{-j\pi n}{2}} = \cos\frac{\pi n}{2} - j\sin\frac{\pi n}{2} \qquad (2.19)$$

Substituting (2.19) into (2.18) results in

$$C_n = \frac{\sin\left(\frac{\pi n}{2}\right)}{n\pi}\left[\cos\frac{\pi n}{2} - j\sin\frac{\pi n}{2}\right] \qquad (2.20)$$

What does this mean? It is a complex variable, just like any other, but in terms of just $n$. We know that $n$ is an integer, so $C_n$ has a number of results, in fact theoretically an infinite number, depending on the maximum value we choose for $n$. As we found with their real counterpart, common with most Fourier transforms in the real world, we do not usually need to consider particularly high values of $n$, only sufficient to give us adequate resolution in the result.

For $n=0$ to $n=8$, Table 2-1 shows the frequency, magnitude, angle (or phase) as well as the real and imaginary parts for the Fourier transform result we obtained in (2.20). Another column is included which shows the logarithmic magnitude in decibels relative to one Volt (dBV). Now that we have considerable information about $C_n$ we must decide what use it is to us.

Just like any other complex number, that in (2.20) has a magnitude and an angle for each value of $n$. Alternatively, it has a corresponding real part and a corresponding imaginary part for each value of $n$.

The cosine and sine arguments are always identical and they are multiples of 90° (or $\pi/2$ radian) which leads to two conclusions:

1. For any fixed value of $n$ the phase of $C_n$ will be a multiple of $\pi/2$, in fact $n\pi/2$.
2. For a given value of $n$, the associated complex vector phase is at 90° to the angle it was at for $n-1$ or $n+1$. This is known as a quadrature (or 90°) relationship.

The quadrature relationship is a very useful one which is exploited widely in digital signal processing and many other areas of electrical engineering. In fact we have decided to express the complex value itself in terms of a quadrature relationship: the real and imaginary axes are at 90° and are therefore in quadrature. This is also clear from the real and imaginary columns shown in Table 2-1: notice that the actual value alternates between the real and imaginary axis.

So what are the amplitude units? We chose a pulse amplitude of 1 V in the original waveform which made life easy, but we could just as easily have generalised it to something like $V_{peak}$. This would not have affected the integration as it is a constant and not a function of $t$. The (linear) magnitude units shown in Table 2-1 are therefore also in volts.

**Table 2-1 Some magnitude and angle (or phase) results for the Fourier transform result of the square waveform (2.20)**

| $n$ | **Frequency (kHz)** | **Magnitude (V)** | **Logarithmic Magnitude (dBV)** | **Angle (radian)** | **Real Part** | **Imaginary Part** |
|---|---|---|---|---|---|---|
| 0 | 0 | 0.500 | -6.0 | 0 | 0.500 | 0.000 |
| 1 | 500 | 0.318 | -9.9 | $-\pi/2$ | 0.000 | -0.318 |
| 2 | 1000 | 0.000 | -100.0 | $-\pi$ | 0.000 | 0.000 |
| 3 | 1500 | 0.106 | -19.5 | $-\pi/2$ | 0.000 | -0.106 |
| 4 | 2000 | 0.000 | -100.0 | | 0.000 | 0.000 |
| 5 | 2500 | 0.064 | -23.9 | $-\pi/2$ | 0.000 | -0.064 |
| 6 | 3000 | 0.000 | -100.0 | $-\pi$ | 0.000 | 0.000 |
| 7 | 3500 | 0.045 | -26.8 | $-\pi/2$ | 0.000 | -0.045 |
| 8 | 4000 | 0.000 | -100.0 | $-\pi$ | 0.000 | 0.000 |

Often, the reason we perform a Fourier transform on a periodic voltage against time waveform is to predict what it would look like in the amplitude against frequency plane, such as how it might appear on a spectrum analyzer. A spectrum analyzer is not sensitive to phase so it will not be able to tell us the values of angle (phase) shown in Table 2-1.

Also we have to be slightly careful with the magnitudes. Very often we use a logarithmic vertical scale on the spectrum analyzer if for example we wish to examine power levels in dBV or dBm. We have magnitude columns for both linear and logarithmic (dB) levels here, so the vertical scale of the spectrum analyzer would need to be set accordingly.

To find frequencies corresponding to values of $n$, it is perhaps easiest to return to the basic Fourier series expression shown in (2.7). The argument of the cosine and sine terms, $n\omega_0 t$ for $n=1$ corresponds to the fundamental frequency $f_0$ of the original periodic waveform, since

$$\omega_0 t = 2\pi f_0 t = 2\pi \frac{t}{T} \qquad (2.21)$$

where $T$ is the period of the original waveform. From Figure 2-2, $T = 2 \quad \mu s$, so $f_0$ is 500 kHz. Therefore the higher order discrete frequency components in Table 2-1 are multiples of 500 kHz as shown. $n = 0$ corresponds to zero frequency or DC. From Table 2-1, the DC magnitude is therefore 0.5 V. Looking at the original waveform in Figure 2-2, we can see that the long term mean (for many cycles) is also 0.5 V since the duty cycle is 50%. This is shown more formally as follows, by using the definition for the calculation of a mean $V_m$ , where $V_{pk}$ is the peak value of the waveform, one volt in this case.

$$\begin{aligned} V_m &= \frac{1}{T}\int_{t=0}^{t=T} v\,dt = \frac{1}{2\mu s}\left[\int_{t=0}^{t=1\mu s} V_{pk}\,dt + \int_{t=1\mu s}^{t=2\mu s} 0\,dt\right] \\ &= \frac{V_{pk}}{2\mu s}\left[t\right]_{t=0}^{t=1\mu s} = \frac{V_{pk}}{2} = 0.5 \quad V \end{aligned} \qquad (2.22)$$

This is perhaps intuitive, because we know that the average value of both a sinewave and a cosine wave is zero, provided it is calculated over a large number of cycles. The averages of the instantaneous voltages for each of the positive values of $n$ in Table 2-1 are therefore all zero and the result of summing the Fourier series is 0.5.

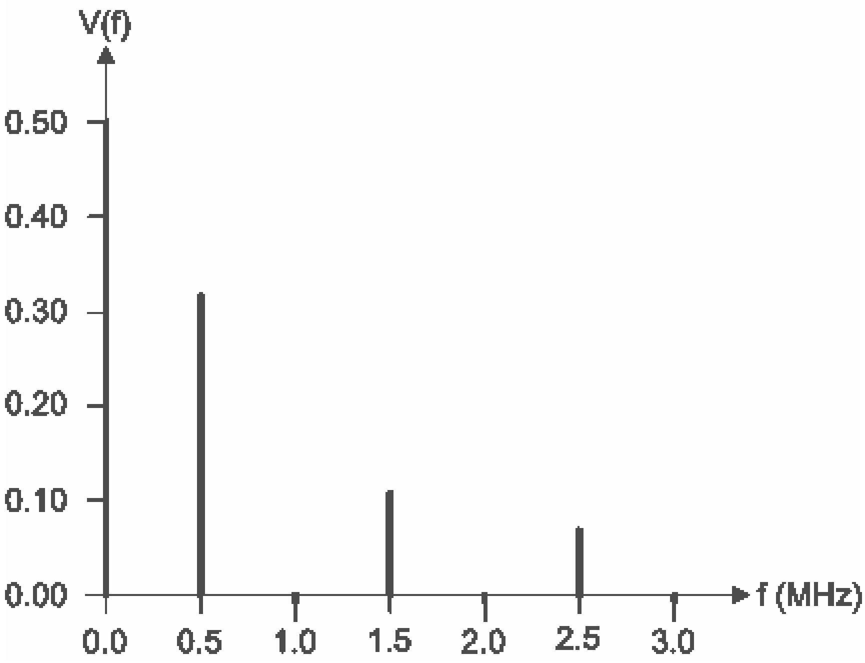


**Figure 2-3 Voltage magnitude against frequency spectrum of the waveform shown in Figure 2-2. (Note that this representation does not provide any phase information.)**

## 2.4 The Complex Fourier Transform of a Regular Pulse Waveform

Often we are confronted with a regular pulse waveform, like that shown in Figure 2-4, which we wish to examine in the frequency domain. This is a more general case of periodic waveform, of which the square wave we investigated in Figure 2-2, is a subset. In this case the pulse width is $\tau$ and the pulse repetition interval, the period of the waveform, is $T$. The frequency of the waveform, also known as the pulse repetition frequency or PRF, $f_{PRF}$ is given by

$$f_{PRF} = \frac{1}{T} \tag{2.23}$$

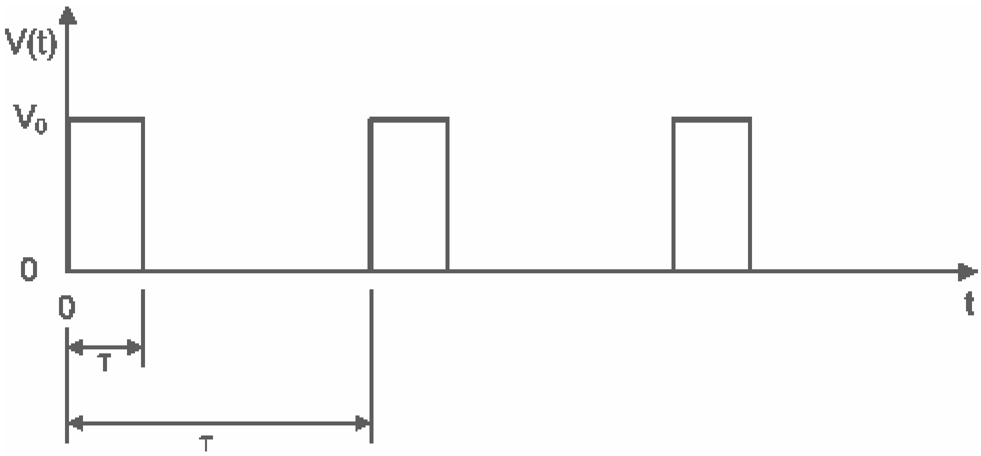


**Figure 2-4 A regular pulse waveform of pulse width τ and pulse repetition interval T**

The peak voltage of the waveform is $V_0$ and we assume that $\tau$ and $T$ are fixed for the duration of the measurement. To convert to the frequency plane we can apply the complex Fourier transform, from (2.13), repeated here

$$C_n = \frac{1}{T}\int_{t=0}^{T} x(t)e^{-jn\omega_0 t}\,dt \tag{2.24}$$

In this generic equation, $x(t)$ just means '$x$, a function of $t$', so it can be replaced by our voltage variable $V(t)$. For $n=0$, the exponential part becomes unity. The integral limits may be shortened to extend from $t=0$ to $t=\tau$, because the pulse is only present for the duration $\tau$ and is zero at other times. Therefore for $n=0$ or the DC component,

$$\begin{aligned} C_0 &= \frac{1}{T}\int_{t=0}^{\tau} V_0\,dt = \frac{V_0}{T}\int_{t=0}^{\tau} dt \\ &= \frac{1}{T}\left[t\right]_{t=0}^{\tau} = \frac{V_0\tau}{T} \end{aligned} \tag{2.25}$$

Evaluating the $n \neq 0$ case is a little more involved because the exponential term no longer evaluates to unity; but conveniently it matches one of the standard integrals of the Naperian constant raised to a complex power of the variable $t$. The value of $x(t)$ is fortunately a constant whilst the pulse is present. This should become clear in the next few lines.

$$\begin{aligned} C_n &= \frac{1}{T}\int_{t=0}^{T} x(t)e^{-jn\omega_0 t}\,dt = \frac{V_0}{T}\int_{t=0}^{\tau} e^{-jn\omega_0 t}\,dt \\ &= \frac{V_0}{T}\frac{1}{-jn\omega_0}\left[e^{-jn\omega_0 t}\right]_{t=0}^{\tau} = \frac{jV_0}{n2\pi}\left[e^{-jn2\pi\tau/T} - 1\right] \\ &= \frac{V_0}{n\pi}\sin\left(\frac{n\pi\tau}{T}\right)e^{-jn\pi\tau/T} \end{aligned} \tag{2.26}$$

In (2.26), Euler has helped us again in converting between the co-sinusoidal (or sinusoidal) complex forms and the exponential complex forms, Using (2.1) and (2.2),

$$\begin{aligned} e^{-j2x} - 1 &= e^{-jx}(e^{-jx} - e^{jx}) \\ &= -2je^{-jx}\sin x \end{aligned} \tag{2.27}$$

We can handle the result of (2.26) just like we did the corresponding square wave result in (2.18). For each positive value of $n$, $C_n$ has an amplitude and phase. It may be easier to expand it in terms of cosine and sine terms. Using the negative exponential version of Euler's identity (2.1), in this case with $x = \frac{n\pi\tau}{T}$, and substituting that for the exponential term in (2.26), $C_n$ can be written in what may be considered a slightly easier form as follows.

$$C_n = \frac{V_0}{n\pi}\sin\left(\frac{n\pi\tau}{T}\right)e^{-jn\pi\tau/T}$$
$$= \frac{V_0}{n\pi}\sin\left(\frac{n\pi\tau}{T}\right)\left[\cos\left(\frac{n\pi\tau}{T}\right) - j\sin\left(\frac{n\pi\tau}{T}\right)\right] \qquad (2.28)$$
$$= \frac{V_0}{2n\pi}\sin\left(\frac{2n\pi\tau}{T}\right) + j\frac{V_0}{2n\pi}\left[\cos\left(\frac{2n\pi\tau}{T}\right) - 1\right]$$

It is interesting to compare the results obtained here for the general pulse waveform to those obtained earlier for the 50% duty cycle waveform. A 50% duty cycle waveform means that the waveform is high for 50% of each cycle, therefore

$$\frac{\tau}{T} = \frac{1}{2} \qquad (2.29)$$

The peak voltage in the 50% duty cycle case was 1 V, so

$$V_0 = 1 \qquad (2.30)$$

Substituting (2.29) and (2.30) into (2.25) and (2.28) yields the results (2.17) and (2.20) respectively.

The result of (2.25) for the zero frequency $n = 0$ or DC case is simply the average voltage of the waveform, over an integer number of cycles. In fact this expression is identical to the definition of the average of such a waveform.

The result for $n \neq 0$ in (2.28) is another complex number for which we can make some inferences by applying the various properties of complex numbers that we have learned over the years. $n = 1$ corresponds to the component at the fundamental frequency, as we saw in (2.23), identical to the frequency of the original waveform $f_{PRF}$.

### 2.4.1 Determining the Complex Parameters of the Fourier Transform

The following trigonometric identities are useful for calculating the complex parameters from (2.28):

$$\sin P \cos Q = \frac{1}{2}\left[\sin(P+Q) + \sin(P-Q)\right] \qquad (2.31)$$

$$\sin P \sin Q = -\frac{1}{2}\left[\cos(P+Q) - \sin(P-Q)\right] \qquad (2.32)$$

By taking the second line from (2.28) and using (2.31) for the real part and (2.32) for the imaginary part, the following results are obtained for the real part of $C_n$, $\mathrm{Re}(C_n)$ and the imaginary part of $C_n$, $\mathrm{Im}(C_n)$.

$$\mathrm{Re}(C_n) = \frac{1}{2}\left(\frac{V_0}{n\pi}\right)\sin\left(2\frac{n\pi\tau}{T}\right) \qquad (2.33)$$

$$\mathrm{Im}(C_n) = \frac{1}{2}\left(\frac{V_0}{n\pi}\right)\left[\cos\left(2\frac{n\pi\tau}{T}\right) - 1\right] \qquad (2.34)$$

The magnitude of (2.28) $|C_n|$ is the square root of sum of the real part squared and the imaginary part squared. After a little manipulation, the result is:

$$|C_n| = \frac{1}{\sqrt{2}}\left(\frac{V_0}{n\pi}\right)\sqrt{1-\cos\left(2\frac{n\pi\tau}{T}\right)} \qquad (2.35)$$

When we refer to magnitude in the sense of (2.35), we do of course mean the *linear* voltage magnitude. In electronics we often use the *logarithmic* magnitude if, for example, we wish to make some measurements in decibels (dB). There are a variety of decibel definitions, for example if we wished to make a measurement in decibels relative to one volt (dBV) the result, say $M_{VdB}$ would be obtained from

$$M_{VdB} = 20\log_{10}|C_n| \qquad (2.36)$$

The argument (or phase) of (2.28) can be obtained easily from the second line, noting that the complex coefficient is actually negative. The following values are obtained for the value in radians ($\phi$) or degrees ($\theta$):

$$\phi = \frac{-n\pi\tau}{T} \qquad (2.37)$$

$$\theta = -\frac{180n\tau}{T} \qquad (2.38)$$

### 2.4.2 Summary of the Fourier Transform Results

Table 2-2 shows two sets of (discrete complex) Fourier transform results, generated using the Microsoft Excel spreadsheet application, in this case version 2002 with the Analysis Toolpak enabled to provide access to the built in complex functions. Both sets have the same pulse amplitude of $V_0 = 1.0 \quad V$. The left set has a duty cycle $\tau/T = 0.5$ and the right set $\tau/T = 0.3$. The rows provide calculated results for $n = 0$ through to $n = 20$. In each case the real coefficient, imaginary coefficient and linear magnitude are obtained from the results of (2.33), (2.34) and (2.35) respectively. The logarithmic magnitude (log mag) columns are the corresponding values of logarithmic voltage magnitudes in the decibel (dB) scale relative to the DC value ($n = 0$). Not many practical systems have have dynamic ranges exceeding 100 dB so log magnitude values less than – 100 dB are displayed as – 100 dB. The arguments are expressed in degrees, using (2.38). The columns headed *Delta Arg (deg)* indicate the relative change in phase for each increment of n.

The results for $\tau/T = 0.5$ or 50% duty cycle also represent a typical digital pulse stream such as might be used to feed a digital modulator of some type in which case the duration of a zero bit is identical to that of a one bit. More precisely we should call this a symbol stream as it is symbols which determine the modulation, dependent on the type of modulation used.

**Table 2-2 Some discrete complex Fourier transform results for $V_0 = 1.0 \quad V$ , $\tau/T = 0.5$ and $\tau/T = 0.3$**

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **Peak V0 =** | | **1.0V** | | | | **Peak V0 =** | | **1.0V** | | | |
| | **Duty Cycle τ/T =** | | **0.5** | | | | **Duty Cycle τ/T =** | | **0.3** | | | |
| **n** | **Real (V)** | **Imag (V)** | **Lin Mag (V)** | **Log Mag (dB)** | **Arg (deg)** | **Delta Arg (deg)** | **Real (V)** | **Imag (V)** | **Lin Mag (V)** | **Log Mag (dB)** | **Arg (deg)** | **Delta Arg (deg)** |
| 0 | N/A | N/A | 0.500 | 0.0 | N/A | N/A | N/A | N/A | 0.250 | 0 | N/A | N/A |
| 1 | 0.000 | -0.318 | 0.318 | -3.9 | -90.0 | | 0.159 | -0.159 | 0.225 | -0.9 | -45.0 | |
| 2 | 0.000 | 0.000 | 0.000 | -100.0 | -180.0 | -90.0 | 0.000 | -0.159 | 0.159 | -3.9 | -90.0 | -45.0 |
| 3 | 0.000 | -0.106 | 0.106 | -13.5 | -270.0 | -90.0 | -0.053 | -0.053 | 0.075 | -10.5 | -135.0 | -45.0 |
| 4 | 0.000 | 0.000 | 0.000 | -100.0 | -360.0 | -90.0 | 0.000 | 0.000 | 0.000 | -100.0 | -180.0 | -45.0 |
| 5 | 0.000 | -0.064 | 0.064 | -17.9 | -450.0 | -90.0 | 0.032 | -0.032 | 0.045 | -14.9 | -225.0 | -45.0 |
| 6 | 0.000 | 0.000 | 0.000 | -100.0 | -540.0 | -90.0 | 0.000 | -0.053 | 0.053 | -13.5 | -270.0 | -45.0 |
| 7 | 0.000 | -0.045 | 0.045 | -20.8 | -630.0 | -90.0 | -0.023 | -0.023 | 0.032 | -17.8 | -315.0 | -45.0 |
| 8 | 0.000 | 0.000 | 0.000 | -100.0 | -720.0 | -90.0 | 0.000 | 0.000 | 0.000 | -100.0 | -360.0 | -45.0 |
| 9 | 0.000 | -0.035 | 0.035 | -23.0 | -810.0 | -90.0 | 0.018 | -0.018 | 0.025 | -20.0 | -405.0 | -45.0 |
| 10 | 0.000 | 0.000 | 0.000 | -100.0 | -900.0 | -90.0 | 0.000 | -0.032 | 0.032 | -17.9 | -450.0 | -45.0 |
| 11 | 0.000 | -0.029 | 0.029 | -24.8 | -990.0 | -90.0 | -0.014 | -0.014 | 0.020 | -21.7 | -495.0 | -45.0 |
| 12 | 0.000 | 0.000 | 0.000 | -100.0 | -1080.0 | -90.0 | 0.000 | 0.000 | 0.000 | -100.0 | -540.0 | -45.0 |
| 13 | 0.000 | -0.024 | 0.024 | -26.2 | -1170.0 | -90.0 | 0.012 | -0.012 | 0.017 | -23.2 | -585.0 | -45.0 |
| 14 | 0.000 | 0.000 | 0.000 | -100.0 | -1260.0 | -90.0 | 0.000 | -0.023 | 0.023 | -20.8 | -630.0 | -45.0 |
| 15 | 0.000 | -0.021 | 0.021 | -27.4 | -1350.0 | -90.0 | -0.011 | -0.011 | 0.015 | -24.4 | -675.0 | -45.0 |
| 16 | 0.000 | 0.000 | 0.000 | -100.0 | -1440.0 | -90.0 | 0.000 | 0.000 | 0.000 | -100.0 | -720.0 | -45.0 |
| 17 | 0.000 | -0.019 | 0.019 | -28.5 | -1530.0 | -90.0 | 0.009 | -0.009 | 0.013 | -25.5 | -765.0 | -45.0 |
| 18 | 0.000 | 0.000 | 0.000 | -100.0 | -1620.0 | -90.0 | 0.000 | -0.018 | 0.018 | -23.0 | -810.0 | -45.0 |
| 19 | 0.000 | -0.017 | 0.017 | -29.5 | -1710.0 | -90.0 | -0.008 | -0.008 | 0.012 | -26.5 | -855.0 | -45.0 |
| 20 | 0.000 | 0.000 | 0.000 | -100.0 | -1800.0 | -90.0 | 0.000 | 0.000 | 0.000 | -100.0 | -900.0 | -45.0 |

There are several ways that the results of Table 2-1 may be represented graphically. For the columns on the left, we have already plotted the linear magnitude against the fundamental frequency coefficient $n$ in Figure 2-3. Although in this section we have been looking at the Fourier transform of a general pulse waveform, the case for a duty cycle of 50% duty cycle waveform is identical to that considered in Section 2.3. One drawback of Figure 2-3 is that it provides no information on the relative phases of the Fourier components at the higher order frequencies. It is simply a two dimensional plot and in fact we can only use the top half of the magnitude axis since magnitude is positive by definition. We saw in Section 2.2 that, in complex frequency analysis, there is some mathematical relevance of the negative frequency axis such as with the results shown in Figure 2-1.

Again, for the results on the left side of Table 2-2, we can gather some very useful information about the behaviour of the amplitude and phase of the Fourier transform products by a three dimensional graph of the type shown in Figure 2-5. Notice that in Figure 2-5 the real and imaginary axes have been replaced by in-phase (I) and the in-quadrature (Q) axes respectively. This is a convention used widely in digital communications, especially for modulation and demodulation. The third (n) axis represents the frequency component of the result of the Fourier transform.

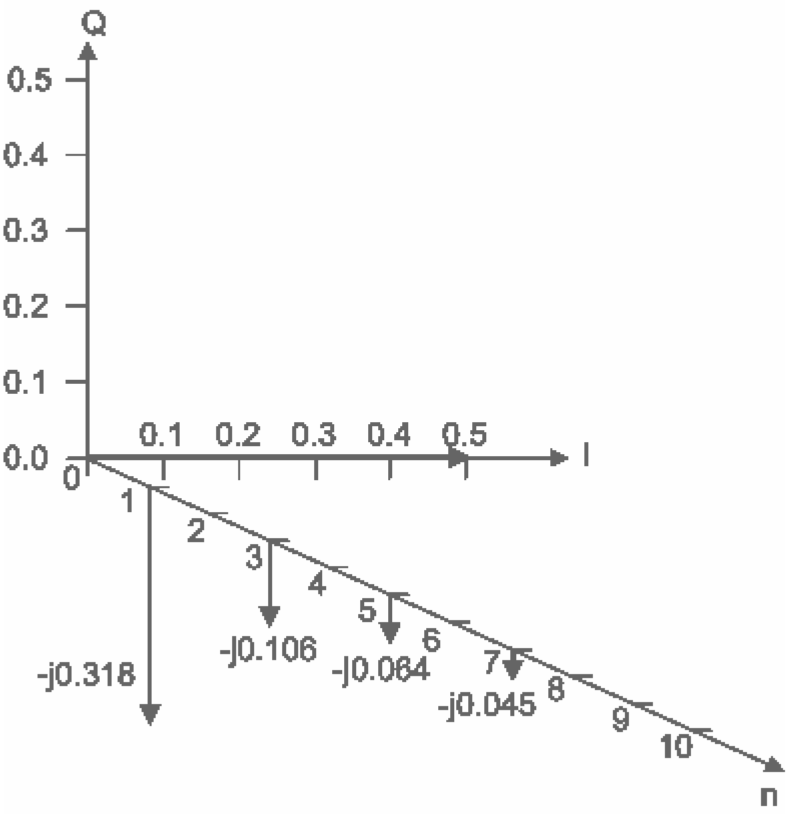


**Figure 2-5 A three dimensional representation of the Fourier transform of a 50% duty cycle pulse waveform with a peak voltage of 1 V from the results on the left side of Table 2-2. Note that all of the components at even values of $n$ are zero.**

Figure 2-5 gives us a vector representation of the fundamental ($n = 0$) plus the first 10 components ($n = 1$ to $n = 10$) of the Fourier transform of the pulse waveform for $V_0 = 1 \quad V$ and $\tau / T = 0.5$ as shown on the left side of Table 2-2. Notice the following important properties:

- the fundamental component is an I (in phase) vector of magnitude 0.5 V;
- there are no other I vectors (in fact mathematically there are I vectors at even values of $n$ , but each has a magnitude of zero);
- the only vectors of finite amplitude are negative Q ones, each at an odd value of $n$, and therefore in quadrature with the fundamental;

### 2.4.3 Orthogonality

One very important relationship shown by Figure 2-5 is that of *orthogonality*. Each of the products shown for odd values of $n$ is at 90° to, and is therefore said to be orthogonal to, the fundamental. The fact that the products at even values of $n$ are all of zero magnitude need not worry us in a practical system as they are not there (indeed not in the real world nor the imaginary world). In fact this can be a very useful property as we shall see.

Many of today's digital communications systems such as those using coded orthogonal frequency division multiplex (COFDM) rely on the transmission of many closely spaced carriers, each of which is digitally modulated with a relatively slow data. COFDM has important advantages where propagation potentially includes significant contributions from multipath. Essentially, the high capacity channel is intelligently split into many slower data streams at the transmit end. Each data stream or channel is then used to modulate one of the (orthogonal) carriers. At the receive end each of the demodulated carriers is recombined again at the receiver to re-create the high capacity channel. In all of these, as the name implies, the Fourier transform products of each modulated carrier are in quadrature to each other carrier. The quadrature property results from the following relationship between the adjacent carrier spacing $f_u$ and the period of the (slow) digital stream used for the modulation of each carrier $T_u$.

$$f_u = \frac{1}{T_u} \tag{2.39}$$

The property also relies on the duty ratio of the digital symbol waveform being 50% ($\tau/T = 0.5$).

If they are in quadrature there is no real component in phase with the fundamental of any other carrier and therefore it cannot cause any interference. The mathematical requirements for orthogonal waveforms will be studied in Section 2.5.

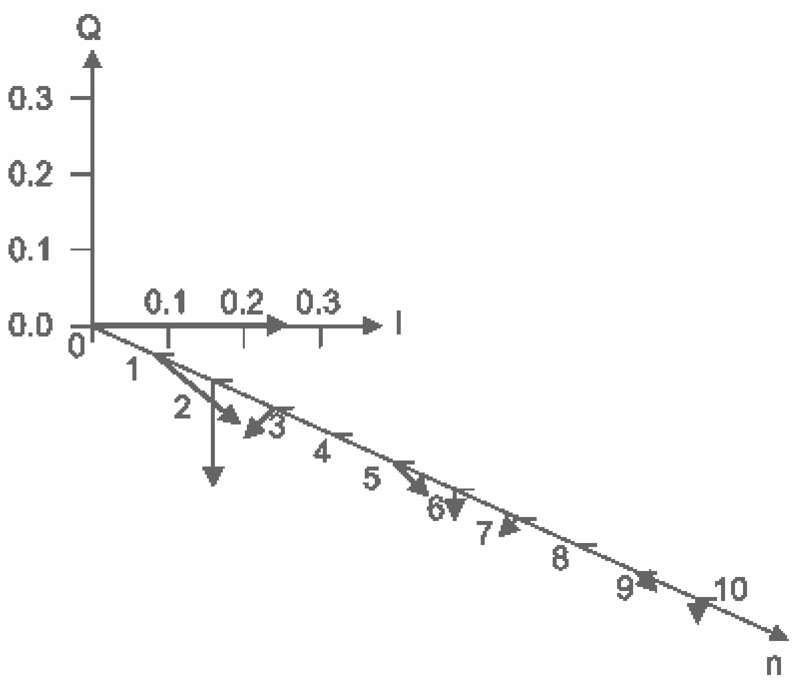


**Figure 2-6 The three dimensional I-Q against Fourier product for $V_0 = 1.0 \quad V$ and $\tau/T = 0.3$**

Figure 2-6 is a similar representation to Figure 2-5 but this time for the right hand columns of Table 2-2, where, again $V_0 = 1.0 \quad V$, but the duty ratio is reduced to 0.3 ($\tau/T = 0.3$).

Immediately we can see that the only perfectly orthogonal components to the fundamental in the range $n = 1$ to $n = 10$ are at $n = 2$, $n = 6$ and $n = 10$. At $n = 4$ and $n = 8$ they are zero, but at other Fourier products component exist which are not orthogonal to the fundamental. This waveform would therefore be unsuitable for a scheme such as COFDM.

## 2.5 The Mathematical Test for Orthogonality

If the waveforms to be tested for orthogonality, expressed as functions of time are $\Psi_p(t)$ and $\Psi_q(t)$ then the following relationships must be met:

- for $p = q$

$$\int_{t=a}^{t=b} \Psi_p(t)\Psi_q(t)^* dt = k \tag{2.40}$$

- for $p \neq q$

$$\int_{t=a}^{t=b} \Psi_p(t)\Psi_q(t)^* dt = 0 \tag{2.41}$$

In these $k$ is a constant and * means complex conjugate. The limits of integration represent a duration of exactly one period of the modulating waveform $T_u$, therefore

$$T_u = b - a \tag{2.42}$$

Considering two adjacent carriers which are separated in frequency by $\omega_u$ where

$$\omega_u = 2\pi f_u = \frac{2\pi}{T_u} \tag{2.43}$$

If the frequencies of the adjacent carriers are $\omega_p$ and $\omega_q$ then

$$\omega_u = \omega_q - \omega_p \tag{2.44}$$

$$q = p + 1 \tag{2.45}$$

In terms of the symbol period

$$\omega_p = \frac{2\pi p}{T_u} \tag{2.46}$$

$$\omega_q = \frac{2\pi q}{T_u} \tag{2.47}$$

The two waveforms may be expressed as:

$$\Psi_p(t) = e^{j\omega_p t} = e^{j\frac{2\pi}{T_u}t} \tag{2.48}$$

and

$$\Psi_q(t) = e^{j\omega_q t} = e^{j\frac{2\pi q}{T_u}t} \tag{2.49}$$

Taking the complex conjugate of (2.49),

$$\Psi_q(t)^* = e^{-j\frac{2\pi q}{T_u}t} \tag{2.50}$$

Taking the easy case first, with $p = q$ and applying (2.40)

$$\begin{aligned}\int_{t=a}^{t=b} \Psi_p(t)\Psi_q(t)^* dt &= \int_a^b e^{j\omega_p t} e^{-j\omega_p t} dt \\ &= \int_a^b e^{j\omega_p t} e^{-j\omega_p t} dt \\ &= \left[t\right]_a^b = b - a = T_u\end{aligned} \tag{2.51}$$

Since $T_u$ is a constant, (2.40) is satisfied.

To verify the condition of (2.41) for $p \neq q$ calls for a little more time and patience with the algebra as follows.

$$
\begin{aligned}
\int_{t=a}^{t=b} \Psi_p(t)\Psi_q(t)^* \, dt &= \int_a^b e^{j\frac{2\pi}{T_u}(p-q)t} \\
&= \frac{1}{j\frac{2\pi}{T_u}(p-q)}\left[ e^{j\frac{2\pi}{T_u}(p-q)t} \right]_a^b \\
&= \frac{1}{j\frac{2\pi}{T_u}(p-q)}\left[ e^{j\frac{2\pi}{T_u}(p-q)b} - e^{j\frac{2\pi}{T_u}(p-q)a} \right] \\
&= \frac{e^{j\frac{2\pi}{T_u}(p-q)b}}{j\frac{2\pi}{T_u}(p-q)}\left[ 1 - e^{j\frac{2\pi}{T_u}(p-q)(a-b)} \right]
\end{aligned} \tag{2.52}
$$

At this point we can substitute (2.42) and use Euler again (2.1) with $x = 2\pi$ , so that

$$
e^{j2\pi} = e^{-j2\pi} = 1 \tag{2.53}
$$

Euler's equations (2.1) are used again and the result of the integration becomes:

$$
\begin{aligned}
\int_{t=a}^{t=b} \Psi_p(t)\Psi_q(t)^* \, dt &= \frac{e^{j\frac{2\pi}{T_u}(p-q)b}}{j\frac{2\pi}{T_u}(p-q)}\left[ 1 - 1^{p-q} \right] \\
&= 0
\end{aligned} \tag{2.54}
$$

Since $p \neq q$ , $p - q \neq 0$ we get no problems with the denominator of (2.54) becoming infinite. However, whatever the values of $p$ and $q$, the value in square brackets becomes zero, so the whole expression becomes zero. The conditions for orthogonality are therefore satisfied.

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