Complex Discrete Fourier Transform (CDFT) and Inverse Complex Discrete Fourier Transform (ICDFT) of a Regular Voltage Against Time Pulse Train

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Complex Discrete Fourier Transform (CDFT) and Inverse Complex Discrete Fourier Transform (ICDFT) of a Regular Voltage Against Time Pulse Train

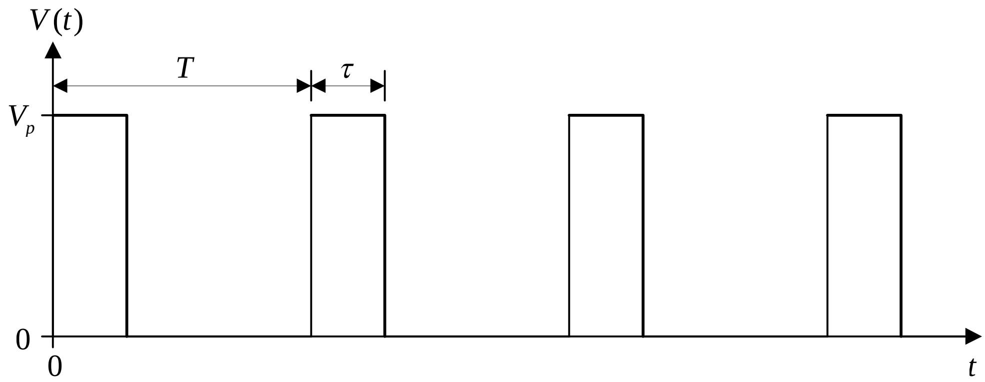
Complex refers to the technique that we will use to perform the Fourier transforms and inverse Fourier transforms: by using complex frequencies, complex number theory and Euler's equations. Discrete means that the transformations will be performed on discrete values, for example sampled values such as we might find at the output from an analog to digital converter (ADC). In this paper, for simplicity we will often generically refer to 'Fourier transforms' or 'inverse Fourier transforms' for which it is understood that we mean CDFTs and ICDFTs respectively.

This paper was written on a Mathcad 14 worksheet with imports from Visio 2003 (diagrams) and Mathtype 6.0 (disabled equations)

You may be wondering if this is anything to do with fast Fourier transforms (FFTs). Today we are not looking specifically at FFTs, but at the general procedures for performing CDFTs on regular voltage-time pulse waveforms and the associated ICDFTs. FFTs is a technique or algorithm for performing CDFTs and ICDFTs quickly if we are using some form of processor or computer. However the complex techniques such as those described must be well understood using 'slow' processing before proceeding to the fast versions.

$j := \sqrt{-1}$ Definition of the imaginary coefficient

Regular Voltage-Time Pulse Waveform



This is a periodic voltage against time (pulse) waveform that we wish to investigate in the frequency domain using the CDFT. It has a pulse width of τ second and a period or pulse repetition interval of T second. The pulse amplitude is V_p volt. For the purpose of the CDFT we will assume that the waveform has been present for a long time.

Note that this is not a modulated waveform, it is a baseband waveform. Nothing has been modulated (yet) and this is the voltage against time waveform as we might measure it on a correctly set up oscilloscope. The duty cycle of a periodic pulse waveform such as this is defined in fractional form as τ/T.

$\tau := 10^{-6}$ Set the pulse width to 1 μs

$T := 10 \cdot 10^{-6}$ Set the pulse repetition interval, or pulse period to 10 μs

$\frac{\tau}{T} = 0.100$ Duty cycle in decimal form (sometimes it is expressed as a percentage)

The Fourier transform means that a periodic waveform such as this may be expressed as a constant (or DC) value plus an infinite series of harmonically related sine waves and cosine waves. This type of series is called a Fourier series. The DC value is equivalent to a zero frequency component. The value of the DC term as well as the amplitudes and frequencies of the respective sine and cosine waves are obtained from the correct interpretation of the results of the CDFT.

The Fourier series representing the pulse waveform is given by the following expression, in complex exponential form:

$$V(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi t}{T}}$$

C_n is the (complex) amplitude of nth complex exponential term.

ω_0 is the angular frequency of the pulse waveform.

If f_0 is the pulse repetition frequency, then $f_0 = 1/T$ and $\omega_0 = 2\pi f_0$.

Infinite Series

The definition of the Fourier series indicates that it comprises an infinite number of terms as the limits of the summation range (n) from -∞ to +∞. One value of n will be at zero frequency, the DC component.
What are the frequencies of the individual components?

From the equation for V(t) above, the phase of the nth term is:

$$\frac{2n\pi t}{T}$$

Therefore, if ω is the angular frequency of the nth term and f is is its (temporal) frequency,

$$\omega t = \frac{2n\pi t}{T}$$

$$2\pi\, ft = \frac{2n\pi t}{T}$$

$$f = \frac{n}{T} = nf_0$$

We know that f_0 is the pulse repetition frequency of the pulse waveform, so the Fourier representation of the waveform comprises the DC component plus sinusoidal and cosinusoidal components at harmonics (integer multiples) of this frequency.

From Euler's equations

$$e^{jx} = \cos x + j\sin x$$
$$e^{-jx} = \cos x - j\sin x$$

From these it follows that

$$\cos x = \frac{e^{jx}}{2} + \frac{e^{-jx}}{2}$$
$$\sin x = \frac{-je^{jx}}{2} + \frac{je^{-jx}}{2}$$

Therefore the series may also be written as

$$V(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi t}{T}} = \sum_{n=-\infty}^{\infty} C_n \left[\cos\left(\frac{2n\pi t}{T}\right) + j\sin\left(\frac{2n\pi t}{T}\right)\right]$$

The coefficients of the terms in the series C_n are given by:

$$C_n = \frac{1}{T}\int_{t=0}^{T} V(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_{t=0}^{T} V(t)e^{-j\frac{2n\pi t}{T}}dt$$

It is very useful to recognise that the coefficients for the positive and negative indices are related by:

$$C_n = C_{-n}{}^{*}$$

where * denotes the complex conjugate.

Sometimes we need to calculate the DC component (n = 0) separately due to 'divide by zero' issues.
So we have to calculate C_n for our particular pulse waveform. This is a relatively straightforward integration so here are a few steps with the result which I think is correct.

$$C_n = \frac{1}{T}\int_{t=0}^{\tau} V_p e^{-j\frac{2n\pi t}{T}} dt = \frac{V_p}{-j2\pi n}\left[e^{-j\frac{2\pi nt}{T}}\right]_{t=0}^{t=\tau}$$

$$= \frac{V_p}{n\pi} e^{-j\frac{n\pi\tau}{T}} \left[\frac{e^{j\frac{n\pi\tau}{T}} - e^{-j\frac{n\pi\tau}{T}}}{2j}\right]$$

$$= \frac{V_p}{n\pi} e^{-j\frac{n\pi\tau}{T}} \sin\left(\frac{n\pi\tau}{T}\right)$$

This is the general expression for our particular voltage against time waveform to calculate C_n.

We cannot literally use limits of infinity because the processing would take forever. Usually we can find suitable finite limits that give us sufficiently accurate results within a reasonable processing time. We will take them from -100 to +100.

sum_start := −100 The start index value.

sum_stop := 100 The stop index value.

In Mathcad, the ORIGIN variable sets the lower limit used for the indices of the matrices. It is not essential, but a logical choice might be sum_start to make these match the actual index values chosen.

ORIGIN := sum_start

n is a range variable with a step of 1 covering the chosen index range

n := sum_start .. sum_stop

n will be split about zero to avoid a divide by zero error and the index = zero case treated separately.

n_plus := 1 .. sum_stop

n_minus := −1 .. sum_start

For example, a peak voltage of 1V.

Vp := 1

Calculate the special case (DC component) for n=0.

$$C_n = \frac{1}{T}\int_{t=0}^{T} V(t)e^{-j\frac{2n\pi t}{T}}dt$$

$$n = 0$$

$$C_0 = \frac{1}{T}\int_{t=0}^{\tau} V_p dt = \frac{V_p}{T}\left[t\right]_0^{\tau} = \frac{V_p \tau}{T}$$

$C_0 := \frac{Vp \cdot \tau}{T} = 0.100$ Calculate C_0 manually, otherwise we get a divide by zero error.

Now calculate the coefficient values for positive indices only starting at 1, up to n_plus.

$$C_{n\_plus} := \left(\frac{Vp}{n\_plus \cdot \pi}\right) \cdot e^{\left(-j \cdot n\_plus \cdot \pi \cdot \frac{\tau}{T}\right)} \cdot \sin\left(n\_plus \cdot \pi \cdot \frac{\tau}{T}\right)$$

Calculate the negative indices by complex conjugates.

$C_{-n\_plus} := \overline{C_{n\_plus}}$ Instead of a star, Mathcad uses a horizontal bar to denote complex conjugate. Notice the negative index and complex conjugate

At this point every element (index value) of C_n should be filled with a complex number. These represent the coefficients of the terms in the Fourier series. Just to confirm, here are some examples written from the array.

$C_3 = 0.050 - 0.069j$

$C_{-3} = 0.050 + 0.069j$ As expected, C_{-3} is the complex conjugate of C_3.

$C_{-23} = 6.581 \times 10^{-3} + 9.058j \times 10^{-3}$ Another example.

The new array D will have an extra dimension to give some extra columns for derived values like frequency, real part, imaginary part etc. By convention, the first index may be considered to be the row and the second index the column of the array, if represented in tabular form.

$D_{n,0} := C_n$ Complex values into column 0.

$D_{n,1} := \frac{\left(\frac{n}{T}\right)}{10^6}$ Frequency MHz, column 1

$D_{n,2} := \mathrm{Re}(C_n)$ Real part, column 2

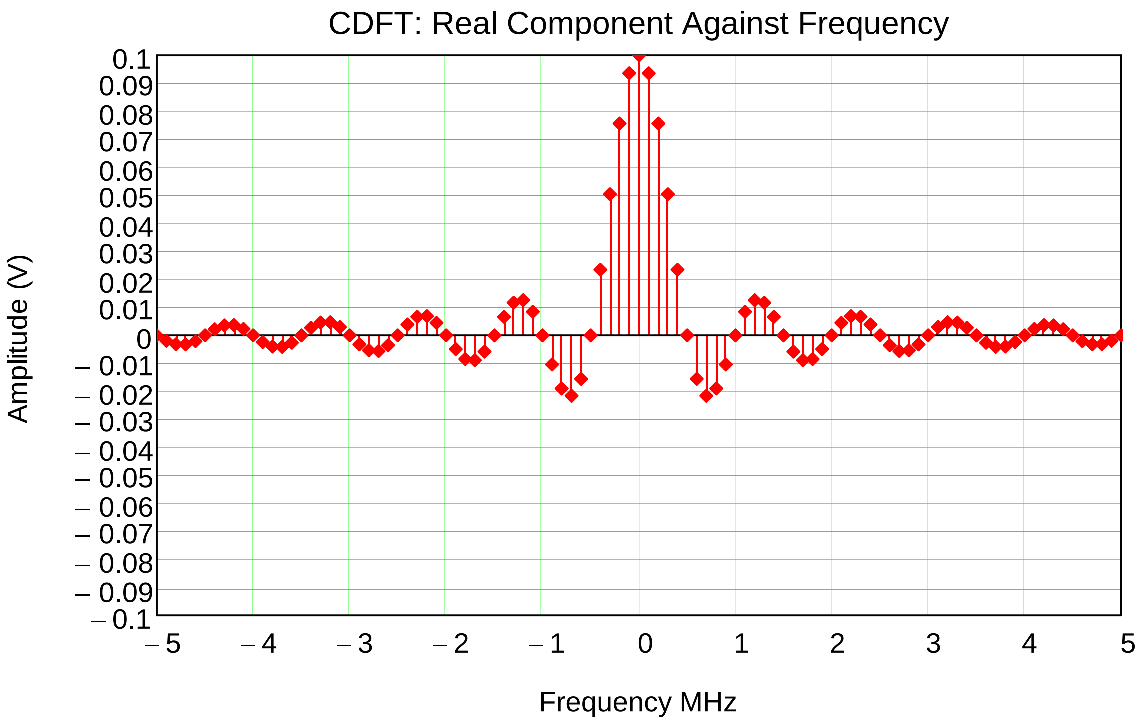
$D_{n,3} := \mathrm{Im}(C_n)$ Imaginary part, column 3.

$D_{n,4} := \frac{180}{\pi}\arg(C_n)$ Angle in degrees, column 4.

$D_{n,5} := |C_n|$ Linear magnitude, column 5.

$D_{n,6} := 20\cdot\log(|C_n|)$ Logarithmic magnitude in dBV, column 6.

The following graphs show these results. Firstly, here are the real components against frequency.



This graph is scaled against frequency and comprises a number of lines, one at each component frequency of the Fourier series, the result of the Fourier transform of the pulse waveform. This is also known as the frequency spectrum of the real components, The spacing of the lines is the reciprocal of the period of the pulse waveform, 1/T. In this case, each line represents the real component of the associated phasor which can of course be positive or negative. Note that the lines are not joined because there are no signals of any type between the lines representing these discrete frequencies (we have assumed that there is no noise present). Notice the similarity of the envelope to a sinc (sinx)/x type function and its symmetry about the vertical axis. Theoretically, the frequency components extend to infinity in both directions because the Fourier series is infinite, from negative infinity to positive infinity, but remember that in this case we used index values ranging from -100 to +100. When performing the transform discretely we need to set a limit to the number of frequency components. The higher the order of the Fourier components, the smaller their amplitude. In a practical system there will be a bandwidth limit above which the products will actually be below the noise floor and may therefore be ignored. It would only be necessary to consider the order of the products up to the point at which they drop below the noise.

Complex Frequency

The reason for using complex (positive and negative) frequencies instead of real ones is to simplify the process of performing the Fourier transforms and inverse Fourier transforms. However it does mean that we have to understand complex frequencies, complex number algebra, Euler's equations and how to use them in the context of periodic (in this case) and aperiodic voltage against time waveforms. Once we achieve reasonable skills in these areas we can handle the three fundamental parameters of sine waves and cosine waves simultaneously: amplitude, phase and frequency.

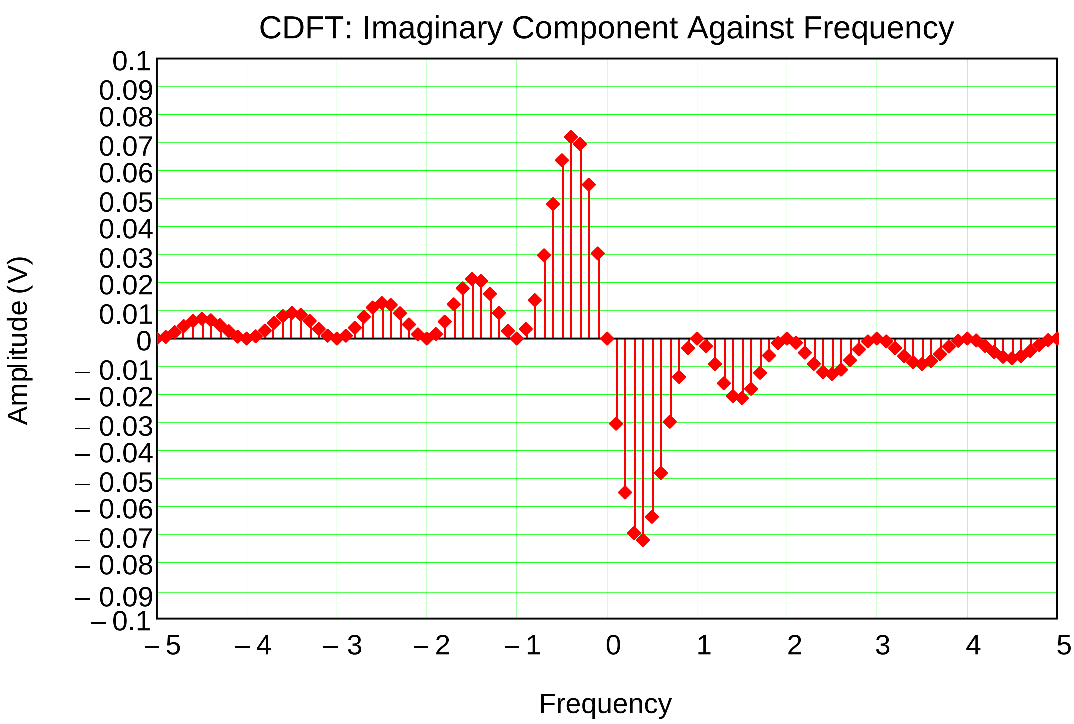
The components of negative frequency are only there to be mathematically correct and to take care of the complex arithmetic. Whenever we need to do anything with the signals in the real world we only consider the positive frequency components.

The example of the pulse signal chosen was a baseband signal waveform which means that it contains components that go all the way down to zero frequency or DC. The DC or zero frequency component can be seen clearly from the graph. In fact, from looking at the voltage time waveform you can see that it is positive going only so must average out to a DC voltage equivalent to the one calculated at zero frequency (n=0). The DC or mean voltage can also be calculated by averaging the area under the voltage against time waveform. Remember that the waveform has not (yet) been used to modulate anything such as an RF carrier using any form of digital modulation such as QPSK or 16QAM.

Real World Interpretation

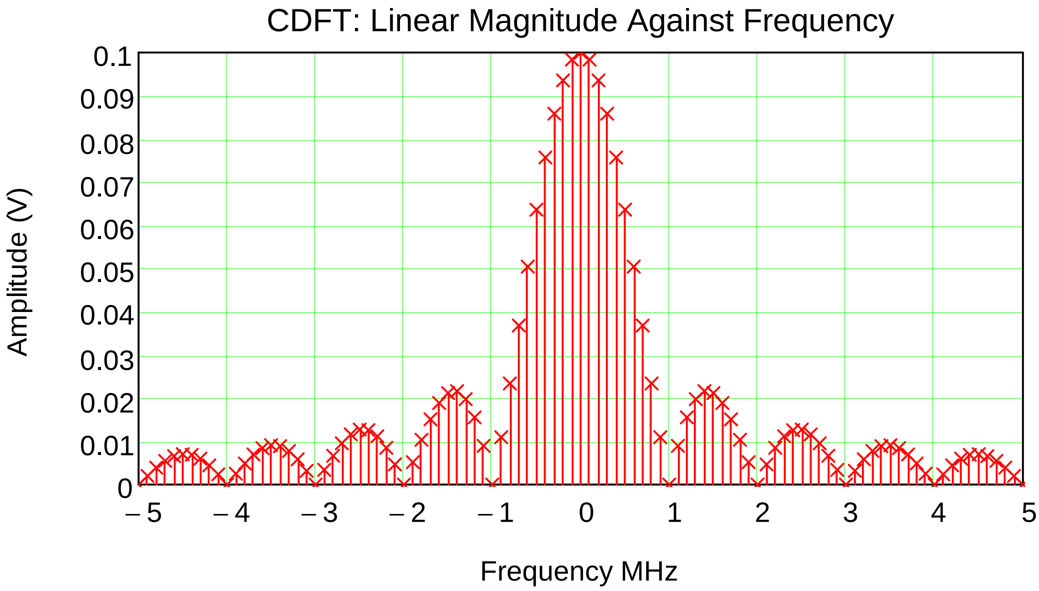
Not to be confused with 'real number interpretation'. Components at negative frequencies do not exist in the real world and cannot be directly measured with a spectrum analyzer or similar instrument. They exist in our complex mathematical mind only. If we wish to examine the spectrum on a spectrum analyser we expect to see only those components in the positive frequency region.

The following graph shows the imaginary components against frequency.



This graph shows the imaginary component against frequency. The same comments apply that were used for the real components. The envelope function in this case is odd (asymmetric about the vertical axis).

The following graph shows the linear magnitudes of the phasors against frequency. The magnitude of each phasor is is of course the square root of the sum of the squares of its real component and imaginary component.



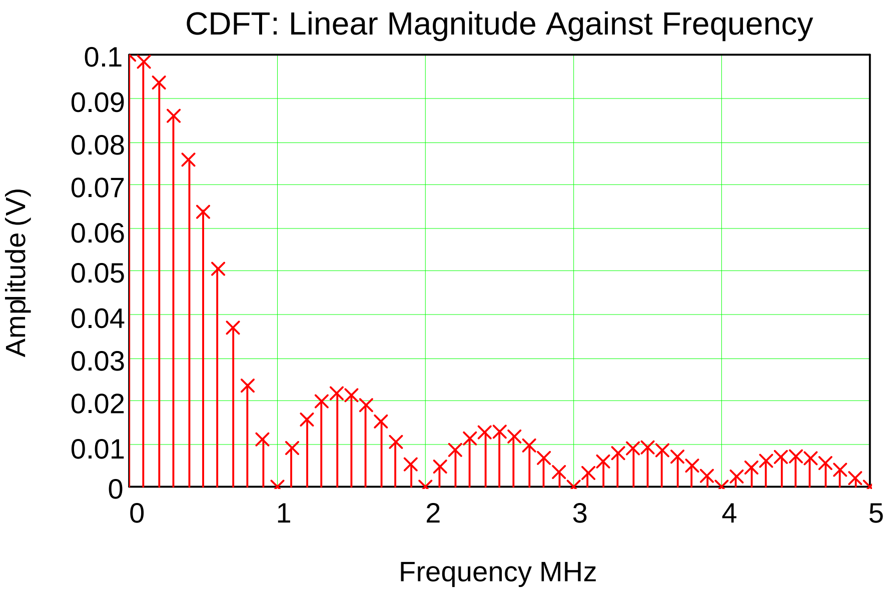
The linear magnitude is the length of the phasor for each frequency component. The frequency spacing of the individual components is again 1/T and the positions of the first and subsequent nulls are multiples of 1/τ from zero frequency, in this case 1 MHz intervals.

Linear Magnitude and Logarithmic Magnitude

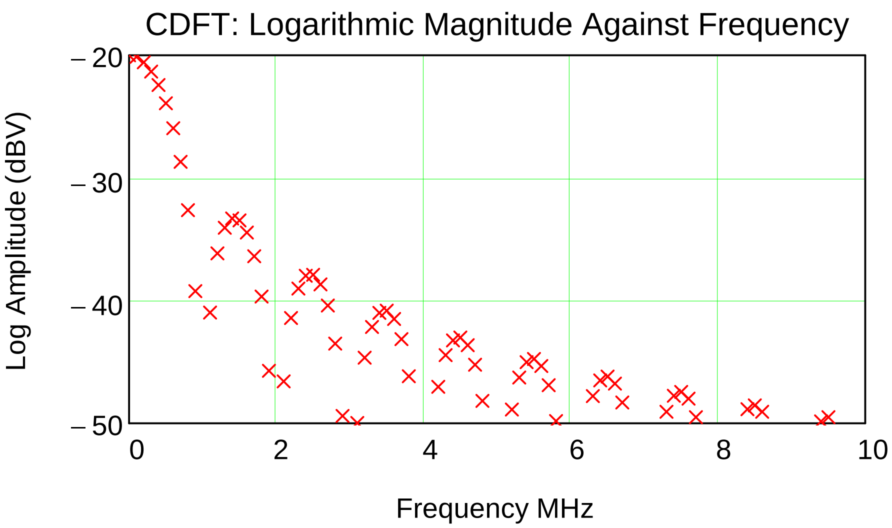
I like to deliberately use the adjectives linear or logarithmic to describe magnitudes in order to distinguish between values expressed as a voltage (or current) such as here, a linear value, and those expressed in a logarithmic unit such as decibels relative to 1 volt (dBV).

Occupied bandwidth

As mentioned above, in the real world we need only consider the positive frequency components of the spectrum. These are shown in the following graph.

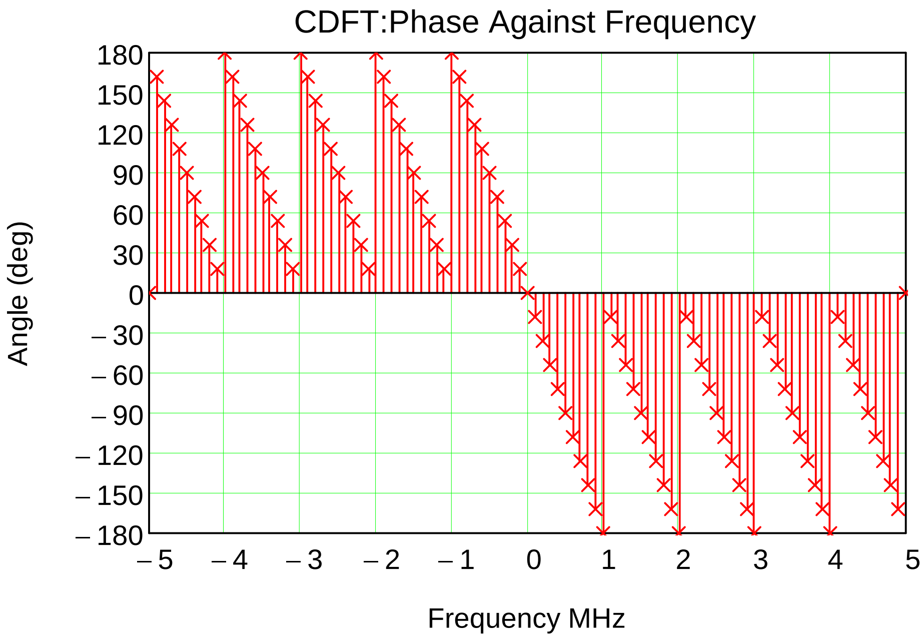


The following graph is the logarithmic magnitude expressed in decibels relative to 1 volt (dBV), again for the real world (positive) frequency components. This is the type of scaling often used on a spectrum analyzer and is useful to display a very wide range such as high level signals in the presence of relatively low levels of noise. To display the individual products accurately the resolution bandwidth (RBW) of the spectrum analyzer must be sufficiently narrow to resolve them.



Phase

The following graph shows the instantaneous values of phase (converted to degrees) for each of the phasors again against frequency. We have adopted the usual convention of positive angles being measured counter clockwise relative to the real axis. Also note that the phase is 'wrapped' at +180 and -180 degrees. This means that once the phase value is less than -180 degrees or greater than +180 degrees, the remainder phase only is plotted. For example, for the phasor at 1.1 MHz, the phase is -198 degrees, the remainder of -18 degrees relative to -180 degrees being plotted.



Notice how there is a linear relationship between the phase angles and frequency.

Inverse Complex Discrete Fourier Transform

We have performed the Fourier Transform to obtain the frequency domain information (frequency spectrum) from the time domain information (a periodic voltage against time waveform). The one dimensional array that we called C_n contains the frequency (derived from n), amplitude and phase information (derived from the complex contents). You will recall that the Fourier series actually contains an infinite number of terms but we have limited that to a number that we can handle easily. The original voltage against time waveform was defined as:

$$V(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2n\pi t}{T}}$$

We simply need to apply this equation using suitable limits of n. The period of the voltage against time waveform is T. This may be determined from the reciprocal of the frequency spacing measured in the frequency domain. We may measure this for example using a spectrum analyzer. If the frequency spacing of the products in the frequency domain was δf, then:

$$T = \frac{1}{\delta f}$$

The following is a Mathcad function for the inverse Fourier transform, expressed as a function of time x(t). The limits of the summation cannot exceed those used in the original (forward) transform because they have not been generated in the first place. In this case they go from -100 to +100. We know that the exact Fourier transform limits go from -∞ to +∞.

$$x(t) := \sum_{n=-100}^{100} \left(C_n \cdot e^{j \cdot n \cdot 2\pi \cdot \frac{t}{T}} \right)$$

We have to create some time scaling for eventual plotting of this waveform, as follows:

$tstart := 0$ The start time in seconds.

$tend := 20 \cdot 10^{-6}$ The end time in seconds

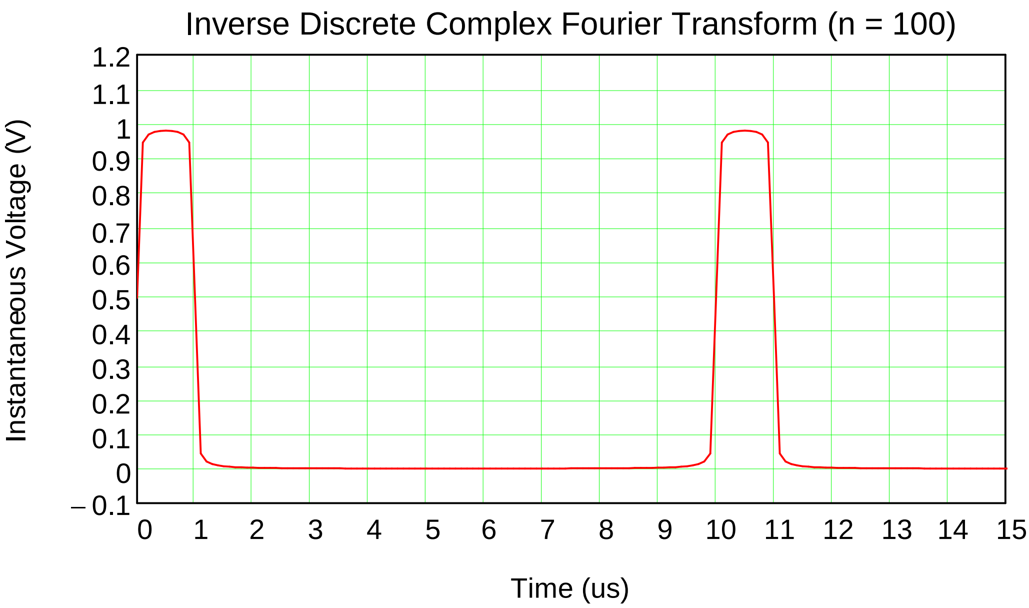
$tinc := .1 \cdot 10^{-6}$ The increment time used in seconds

$tsteps := trunc\left[\frac{(tend - tstart)}{tinc}\right]$ Total number of time steps, truncated as we do not want fractional parts

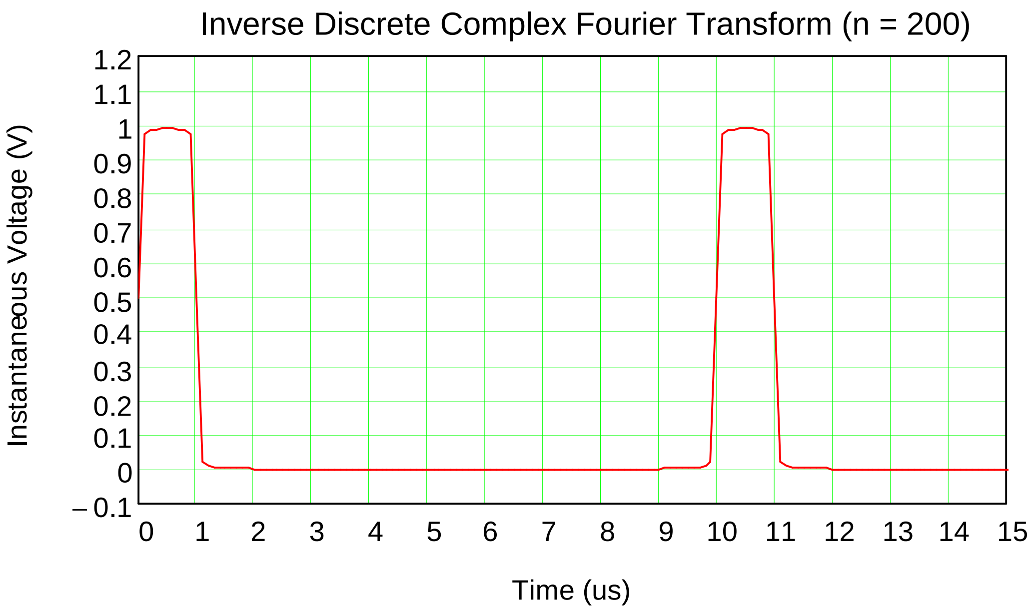
$tindex := 0 .. tsteps$ Range variable to handle the timing (timing index)

$tval_{tindex} := tstart + tindex \cdot tinc$ Range variable of actual time intervals in seconds

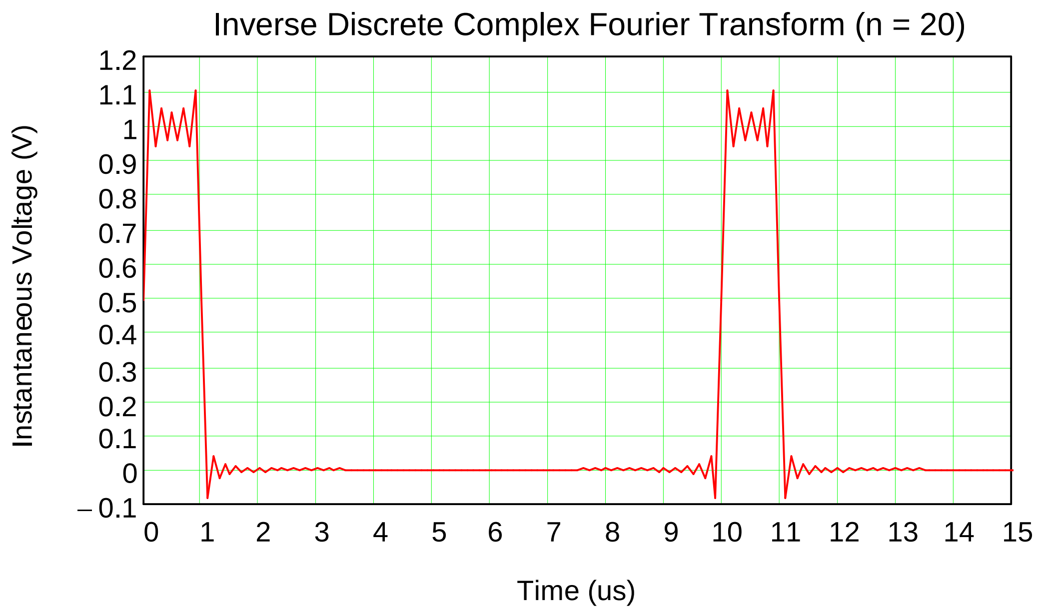
Here is the result. (The time axis has been converted to microseconds)

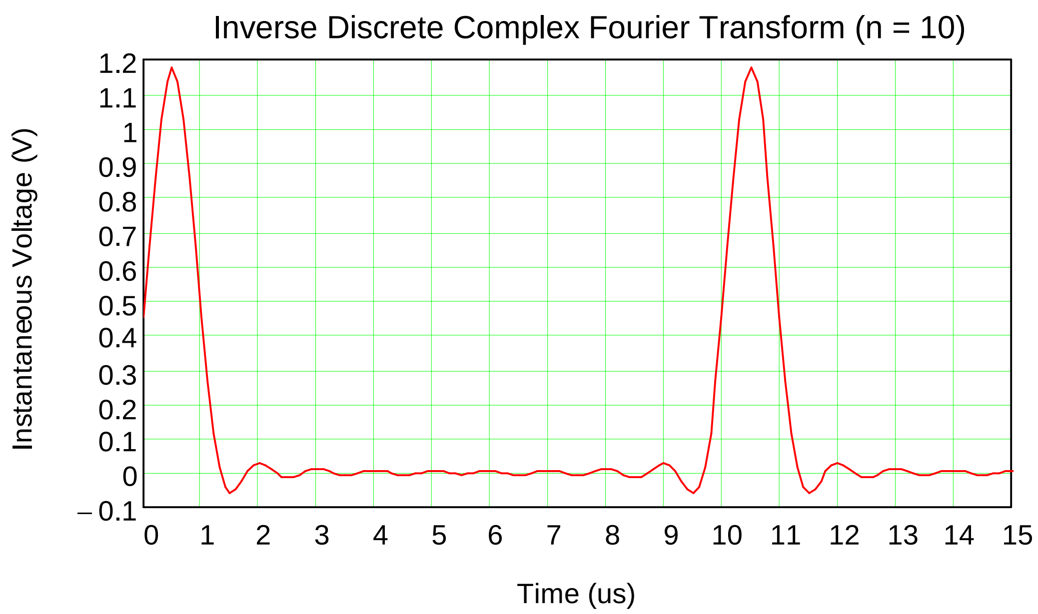


The following examples are the results of the same inverse transform but with using the values of n as shown.



This example with n = 200 was of course only possible if the forward Fourier transform was performed with n = 200 or more.

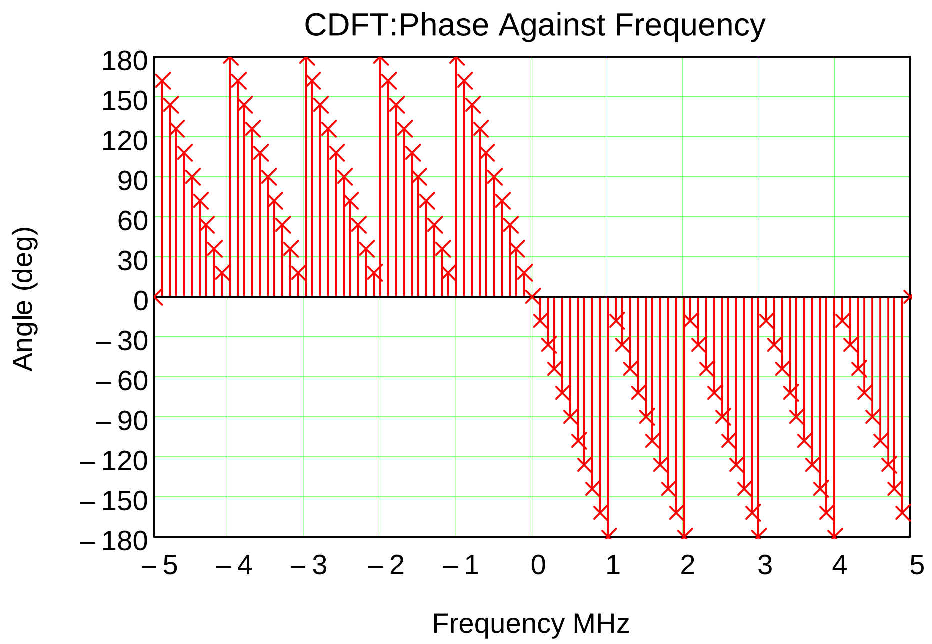




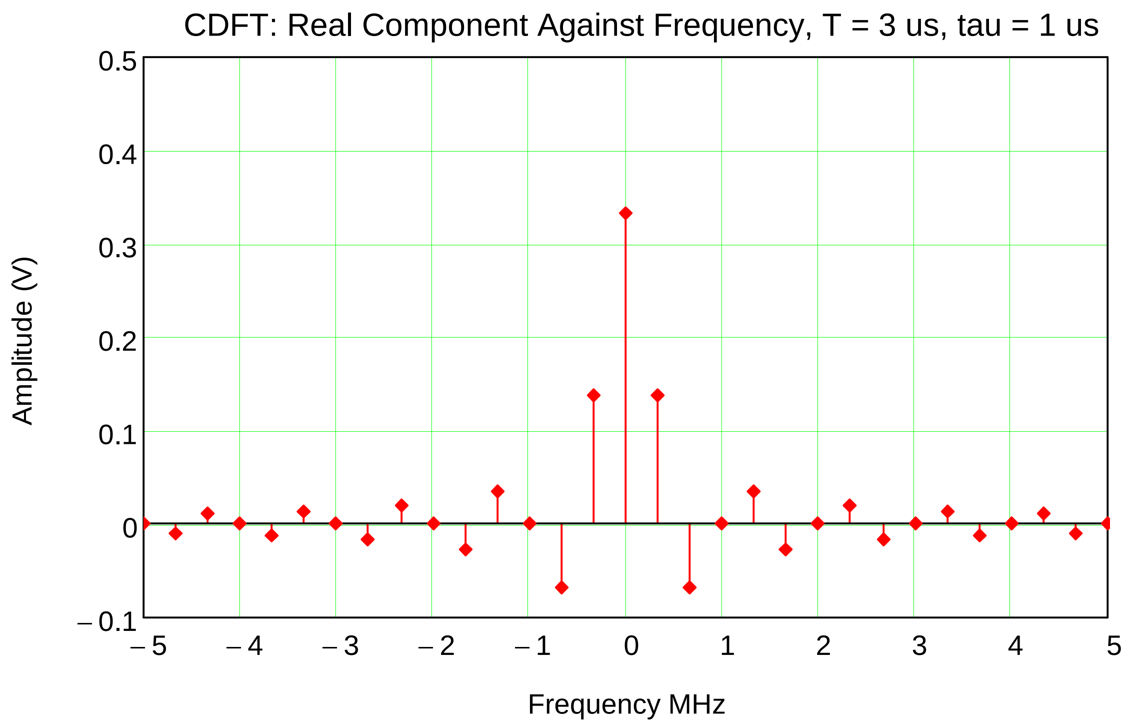
We can see from these results that the closer replica of the original pulse waveform is formed with the largest value for n.

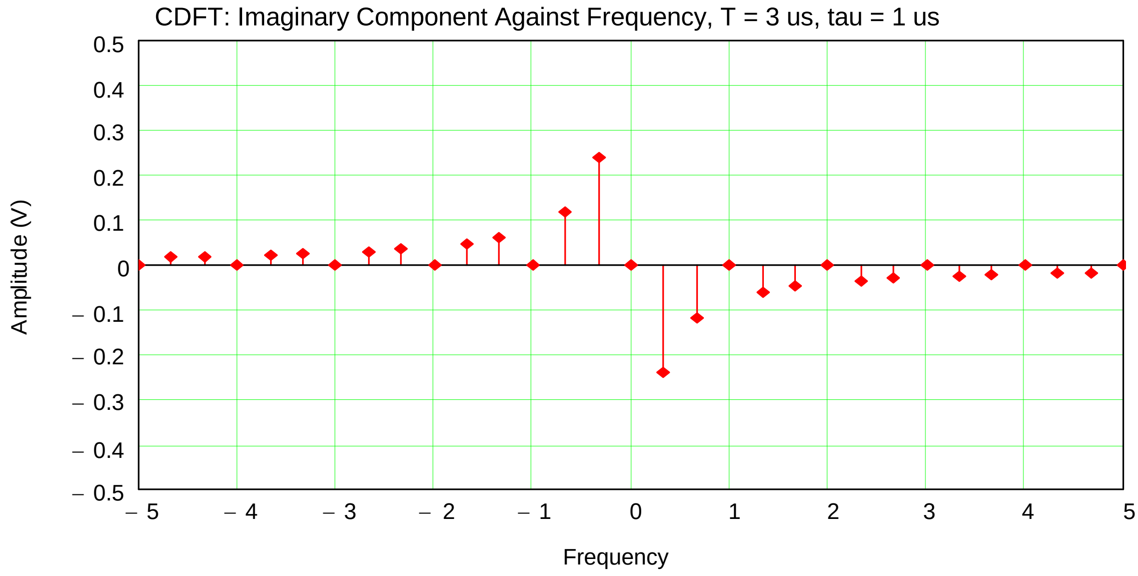
Orthogonal Frequency Spectra

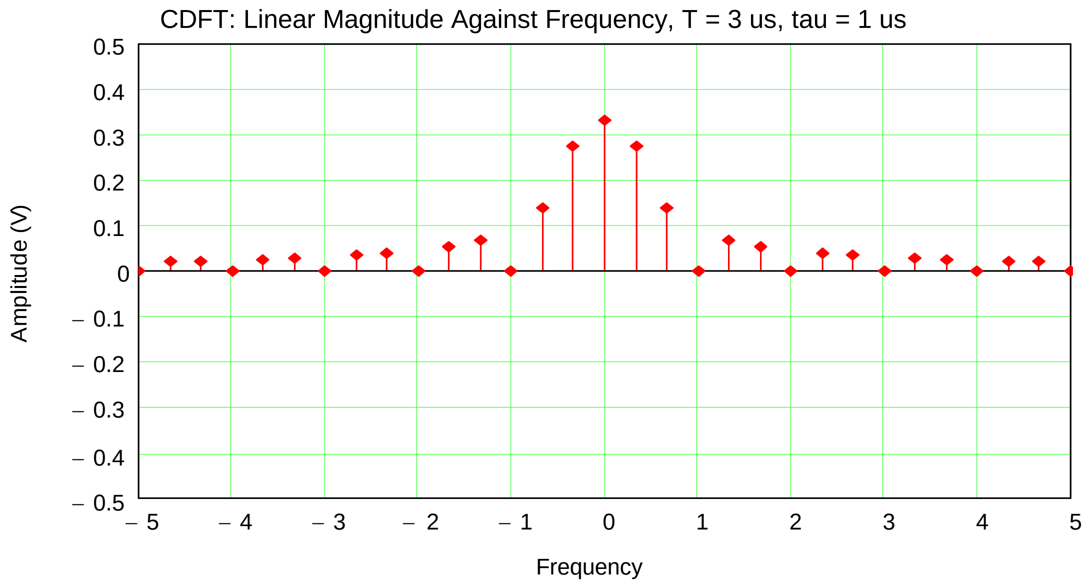
Orthogonal means 90 degrees, in this case referring to the difference in instantaneous phasor angle between adjacent frequency spectrum products or sub-carriers. These have some very useful properties. Firstly we will examine some non-orthogonal sub-carriers. The graph below shows the phase angles of the carriers in the original example for which T = 10 us, τ = 1 (the associated duty cycle being 0.1).

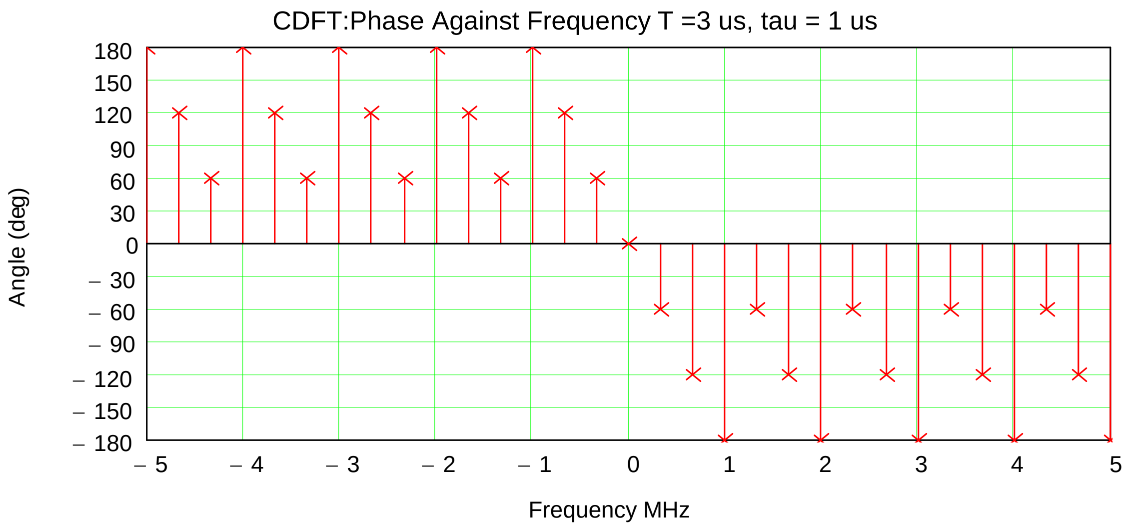


There are 10 linearly spaced values of phase over the frequency range corresponding to the reciprocal of the pulse width (1/τ). As we have established, each frequency product is spaced from the next by the pulse repetition frequency (or the reciprocal of the pulse period). In this case the duty cycle was 0.1. The following plots in the frequency domain are for T = 3 μs and τ = 1 μs, a duty cycle of 0.333.

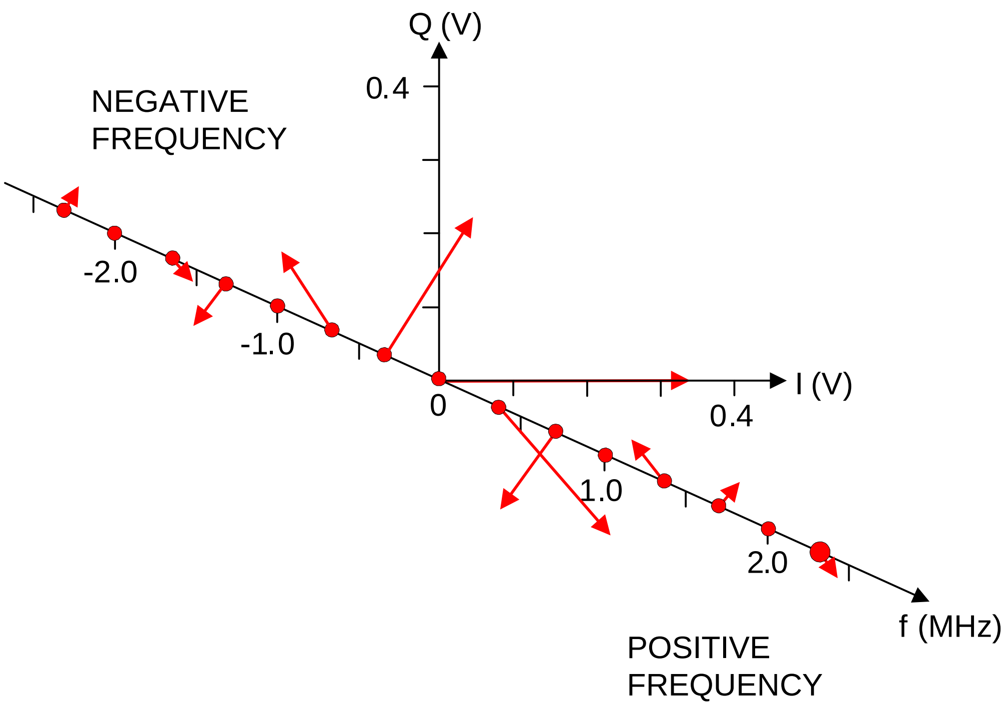



CDFT: Imaginary Component Against Frequency, T = 3 us, tau = 1 us
Amplitude (V)
Frequency


CDFT: Linear Magnitude Against Frequency, T = 3 us, tau = 1 us
Amplitude (V)
Frequency



Sometimes it is easier to visualise the magnitudes and rotations of the phasors if they are are represented in 3 dimensions, perhaps as shown in the example below which uses the mutually orthogonal axes I, Q and f (frequency in megahertz). This taken at an instant in time when the phasor at zero frequency is aligned perfectly with the real or in-phase (I) axis. At this instant, the amplitudes and angles of the other phasors are as shown. We can see from the phase against frequency graph above that, for this waveform, the instantaneous phase differences between adjacent phasors is 60 degrees. The angles again increase in the negative frequency direction and reduce in the positive frequency direction, again taking counter clockwise rotation of the phasor as positive.

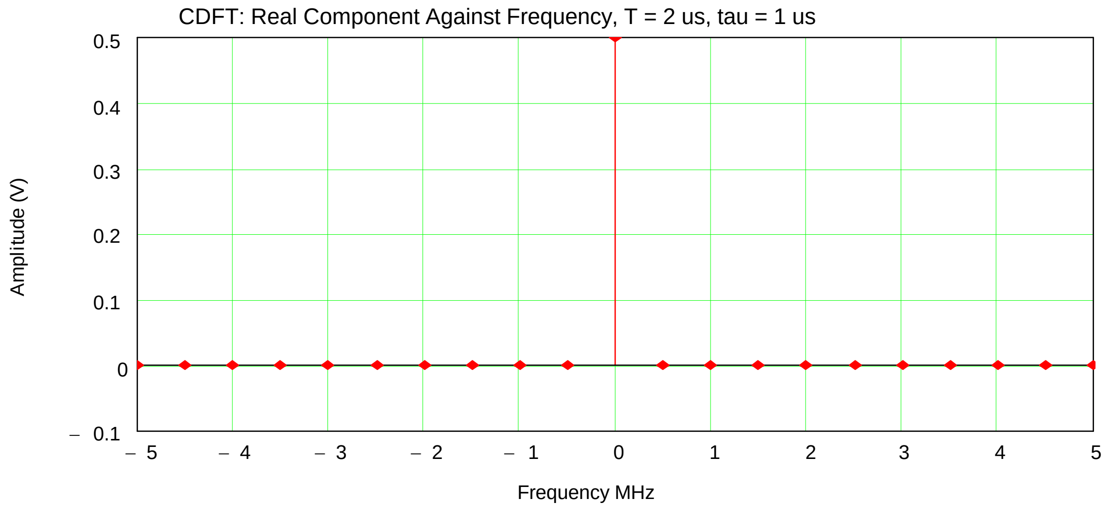


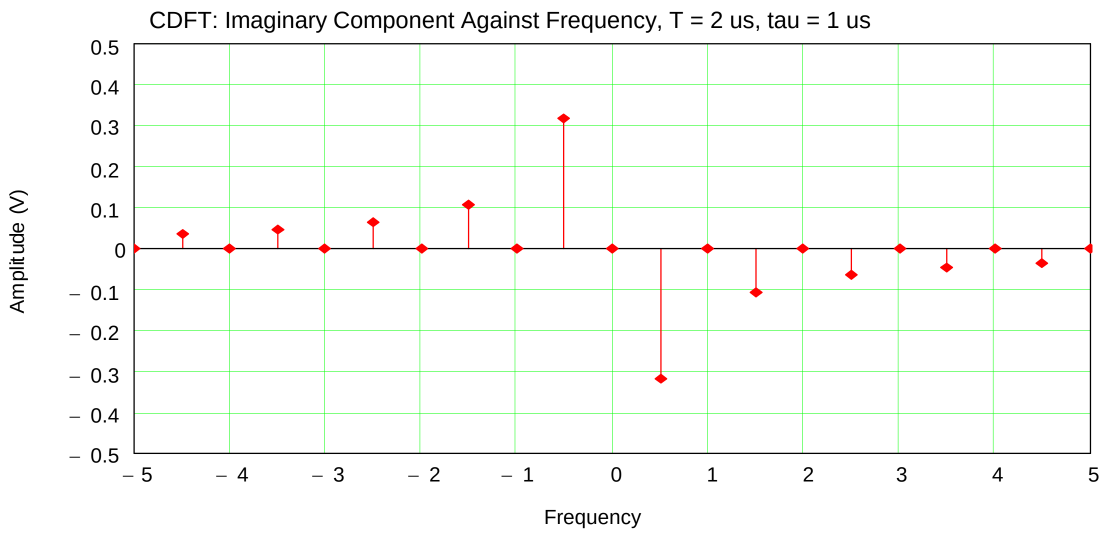
Orthogonal Subcarriers

A very important case occurs when the duty cycle of the original voltage against time waveform is 0.5

or 50%. Under this condition, the instantaneous phase angle between adjacent carriers (or sub-carriers) is 90 degrees, so they are orthogonal. Example graphs of the results of the Fourier transform of such a waveform (with T = 2 μs and τ = 1 μs) are shown below. With such a duty cycle the voltage against time waveform is like a repetitive stream of data comprising 101010..., where 0 is defined as zero volts and 1 is defined as a voltage V_p. That is to say that the time for the 0 state is identical to the time for the 1 state and in this example the duration of each bit is 1 μs, equivalent to a data rate of 1 megabit per second (1 Mbit/s). The information content of such a stream of data would be zero and not very useful. However, the 1010... stream of data after applying the Fourier transform, is found to occupy the largest bandwidth (or be 'spread' across the largest bandwidth). Therefore this is like a worst case condition that we can base the design of a digital communications channel on. We can then go ahead and send useful data streams over the link instead of just 101010.... The way the associated frequency spectrum is spread out over frequency will depend on what data is actually sent but it will occupy less spectrum than the 101010... case.

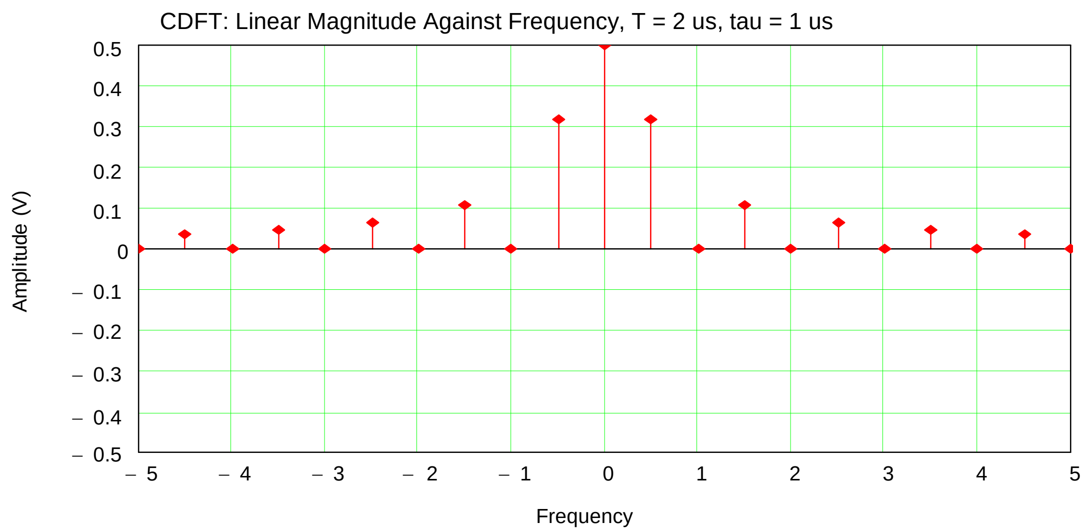
With orthogonal sub-carriers, the phase of every second carrier will either be exactly in phase (or exactly 180 degrees out of phase) with the original carrier at zero frequency. However, as the graphs of real components below show, the magnitudes of each of these is zero.



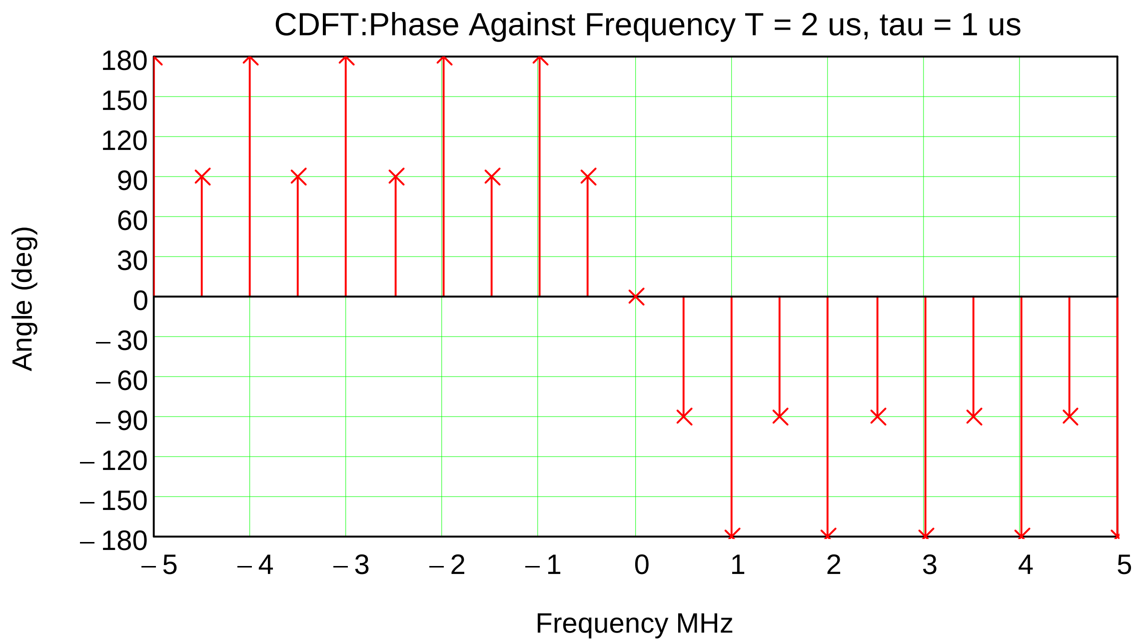


In fact, when we look at the I and Q components of each of the sub-carriers, we find that, in each case there is either zero I component or zero Q component. Furthermore, with the exception of the zero frequency case, the I component magnitudes are all zero. The Q component magnitudes are finite at odd values of n.

The following plot is one of linear magnitude against frequency, for completeness.



The following plot of phase against frequency confirms that, at the instant considered, the phase of each carrier is at 90 degrees (or orthogonal) to the adjacent one.

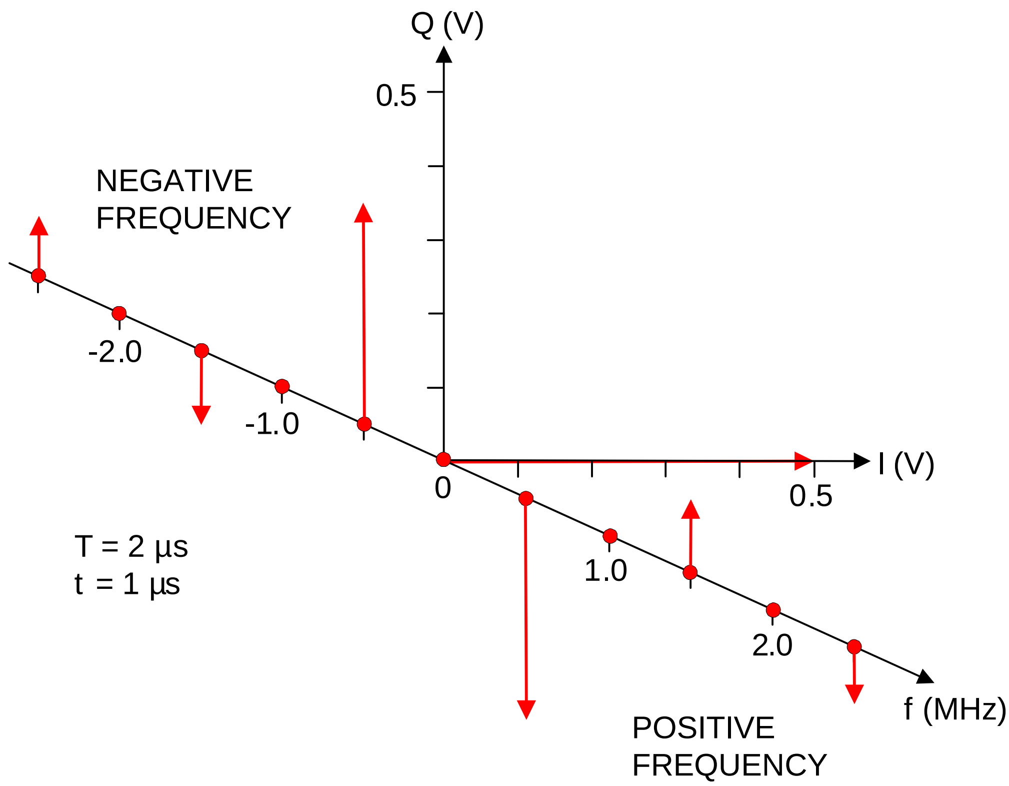


The following diagram is another representation in 3 dimensions of the phasors for the orthogonal case. Remember again that this is at the particular instant when the phasor representing the zero frequency carrier has no imaginary component.

It clearly shows some interesting properties:

There are no I components other than that of the zero frequency component itself. (Actually theoretically there are I components, at 1 MHz and 2 MHz for example, but their magnitudes are zero).

The only other finite magnitude components, at 0.5 MHz, 1.5 MHz and 2.5 MHz for example, are actually Q components so are therefore orthogonal to the zero frequency component.



This orthogonal property is exploited in many communication systems that deliberately use multiple closely spaced sub-carriers instead of one main carrier. Examples of these are:
Orthogonal Frequency Division Multiple Access (OFDMA) as used in the WiMax standard (IEEE 802.16) and the WiFi standard (IEEE 802.11).

Coded Orthogonal Frequency Division Multiplex (COFDM), as used in the digital audio broadcast (DAB) standard.

Each of these comprises many closely spaced, orthogonally related sub-carriers. This relationship effectively means that a small degree of interference between adjacent carriers can actually be tolerated They would not be at baseband as we considered above, but up-converted to occupy a suitable radio frequency channel. The process of up-conversion would shift the zero center frequency to an appropriate frequency allowed by licensing limitations and the available technologies for antennas and appropriate radio receivers. For example, if the frequency shift was 200 MHz, zero frequency would be shifted to precisely 200 MHz. The sub-carriers would still maintain the orthogonal relationship so the component at +0.5 MHz would be shifted to 200.5 MHz and that at -0.5 MHz would be shifted to 99.5 MHz, and so on. In fact, for the examples given, the required resilience to multipath fading means that the sub-carriers are spaced in the order of kilohertz, rather than megahertz. Individual sub-carriers are therefore the result of relatively slow data. For example, if the period of the data stream was 64 µs (a data rate of 32.250 kbit/s) then the sub-carrier spacing would be the reciprocal of this, or 15.625 kHz.

References

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