Complex Discrete Fourier Transform (DFT) and Complex Inverse Discrete Fourier Transform (IDFT) of a Periodic Arbitary Voltage Against Time Wave

Some examples using Mathcad 14.0 by Chris Angove, January 2010

Reference:
Smith, Steven W.; The Scientist and Engineer's Guide to Digital Signal Processing - Second Edition; California Technical Publishing, P. O. Box 502407, San Diego, California 92150-2407; Chapters 8 and 31. http://www.dspguide.com. ISBN 0-9660176-7-6 (1999)

In Mathcad, ORIGIN sets the minimum index that may be used for matrices, in this case it will be set to zero. This is not essential but is frequently adopted. Often in digital signal processing we prefer to start counting at zero because we often work in powers of 2 with 2^0 being the least significant bit.

$\mathbf{ORIGIN} := 0$ sets the lowest index for Mathcad arrays to zero.

Voltage against time data were generated externally and are imported here using the Mathcad READFILE() function. In this case the data was in the form of two columns from an Excel spreadsheet: in the first column the time in microseconds starting at zero with one microsecond steps, and in the second column, the instantaneous voltage in volts. The number of time steps in this case was 256, or 2^8. Often we take a number of samples of this type which may be expressed by an integer power of 2 for the benefit of the (discrete) fast Fourier transform (FFT). For the FFT, 'fast' refers to the relative speed of execution of the algorithm that performs the transform compared to the 'traditional' algorithm. Here we will look at the traditional approach. The definition of the transform itself is identical whichever algorithm is used.

The time and voltage data are read into a Mathcad array with two dimensions called $InDat_{mn}$, where m and n are the array subscripts which, in this simple two dimensional case, relate to the rows and columns imported from the Excel spreadsheet. Both of these, as we just mentioned, start at zero. The first subscript 'm' is the number of the co-ordinate timestep and the second subscript 'n' can take either of two values 0 or 1 corresponding to columns 0 or 1, containing the time values and the voltage values respectively. Since in this case there are 256 samples, the first subscript starts at zero and ends at 255.

Seems that in Mathcad you have to define the complex

$\mathbf{j} := \sqrt{-1}$ t j.

Read in the time - voltage data

$$\mathbf{InDat} := \mathbf{READFILE}\left[\text{"I:\textbackslash Mathcad\textbackslash DataImport\_1.xls"}, \text{"Excel"}, \begin{pmatrix} 11 \\ 266 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]$$

The column vectors in the READFILE() function define where Mathcad looks for the rows and columns in the Excel spreadsheet.

Just some examples displayed of what was read into InDat...

$\mathbf{InDat}_{0,0} = 0$ row = 0, column = 0

$\mathbf{InDat}_{0,1} = 0.025$ row = 0, column = 1

$\mathbf{InDat}_{255,0} = 255$ row = 255, column = 0

$\mathbf{InDat}_{256,0} = \blacksquare$ row = 256, column = 0, row exceeds the maximum so no value is recognised and display is red

Use the Mathcad rows() function to read the total number of samples, equivalent to the number of rows and define the result as 'N'.

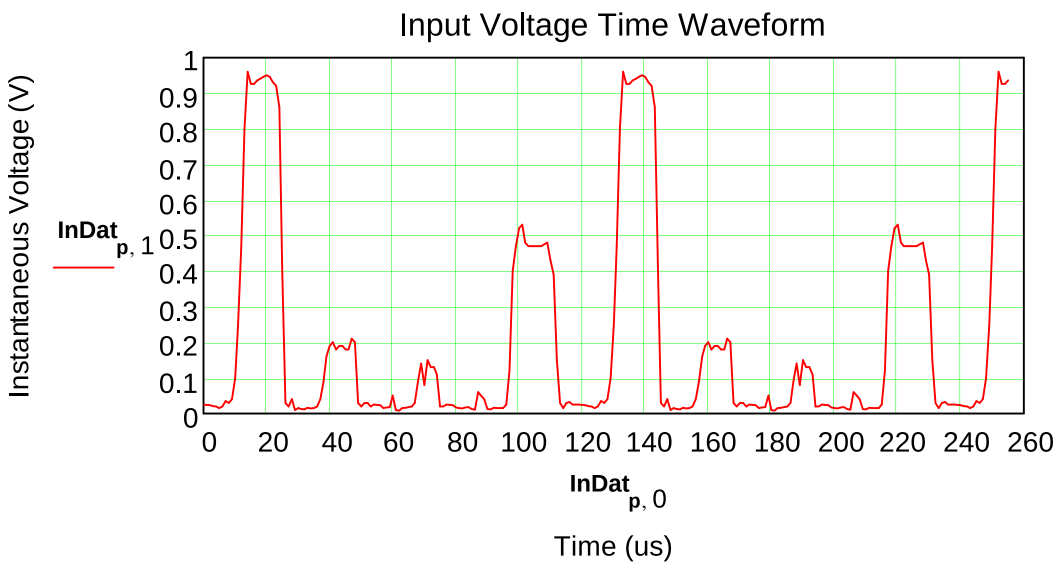
$\mathbf{N} := \mathbf{rows}(\mathbf{InDat}) = 256$

Set up a range variable called p to handle the 256 samples ranging from 0 to 255.

$\mathbf{p} := 0 .. \mathbf{N} - 1$

What does the input voltage time waveform look like?

The following graph is a plot of what was read into the array InDat for column 1 (voltage) against column 0 (time in microseconds)



Just digressing a bit, the input voltage against time waveform resembles that we might have obtained from a short range radar receiver: a number of pulses of various amplitudes. The waveform is clearly periodic, in this case with a period of about 120 μs, also known as the pulse repetition interval (PRI). The large pulse is direct from the transmitter and the smaller pulses, those received from various stationary or slow moving targets at progressively increasing ranges. If it was a radar, we can see that the pulse repetition frequency (PRF), the reciprocal of the PRI of 120 μs, is 8.3 kHz. The pulses are probably a bit fatter than one would normally get and in this case reflected pulses can be resolved up to approximately 100 μs after the transmit pulse. The corresponding distance for a total time of propagation of 100 μs is about 30 km so the range of this radar, allowing for the forward and reflected pulses would be around 15 km.

A total of 256 samples were taken, with 1 μs spacing which we will call deltatime, so

$\mathbf{deltatime} := 10^{-6}$ sample interval in s

The sample frequency fs, is the reciprocal of the sample spacing, so

$\mathbf{fs} := \dfrac{1}{\mathbf{deltatime}}$ sample frequency Hz

As well as discrete functions of the type being considered, Fourier transforms (FTs) may be performed on continuous functions. In these the result is true literally for all values within the range considered, including those between the sample points. In practical systems such as we are considering, it would be impossible to implement an equivalent method for continuous functions because spacing between samples would need to be zero and therefore the number of samples would be infinite and impossible to handle. We have to choose a suitable sample spacing which will both describe the original waveform adequately and not generate too many samples. Also we often need to process the data in real time which can be difficult to do quickly enough if we have too many samples.

After applying the complex discrete Fourier transform (CDFT) to the voltage time waveform like that considered, the result will be a finite set of (voltage against time) cosine waves, sine waves together with a constant value, often called a DC offset. The DC offset is in fact equivalent to the first coefficient of the cosine term for which the frequency is zero. The cosine of zero is of course 1 but the sine of zero is also zero so the first term of the sine series is always zero. These are often called orthogonal sets, the word 'orthogonal' referring to the 90 degrees temporal phase difference between them. Each waveform is mathematically assumed to be infinitely long and the highest frequency waveform is determined by the Nyquist minimum sampling frequency criterion. This may be described in many different ways. One is that a minimim of two samples are necessary to fully describe the frequency and amplitude of a sinusoidal or cosinusoidal waveform. If we choose a fixed sample spacing such as the 1 μs used in this case, the highest frequency which may be represented would be when one wavelength is exactly equal to 2 sample spaces. This frequency would be exactly one half of the sample frequency itself.

A major advantage of the CDFT compared to its real equivalent is that we can exploit the rules of complex algebra and, in particular, Euler's identity which relates exponential (polar) and trigonometrical (rectangular) expressions as follows.

Euler's Identities

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

After performing the CDFT on a set of voltage against time points for example, as we have mentioned, the result will be a set of real components and a corresponding set of imaginary components. These will define the coefficients of the cosine and sine waveforms respectively.

The CDFT X_k of a discrete variable in the time domain, x_n is defined as:

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-j2\pi kn/N}$$

where

- x_n is a discrete variable in the time domain such as discrete voltages
- n is the sample number, starting at zero and ending at N
- N is the total number of samples taken
- k is the index (subscript) of the (complex) result of the Fourier transform

This version of the CDFT is in polar (or complex exponential) form. It can be expressed in rectangular form if desired by applying Euler's equation:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

where

$$\theta = \frac{2\pi kn}{N}$$

so the rectangular form is

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n \left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right]$$

Mathcad allow us to calculate the CDFT directly using the polar form.

Now set up a range variable in Mathcad for the subscript of the CDFT result. Again the subscript starts at zero and ends at N-1. The symbols in Mathcad correspond to those in the FT definition above with the exception that the voltage against time waveform is taken directly from the array called $InDat_{x,y}$ that was filled from the

$$\mathbf{k} := 0 .. \mathbf{N} - 1$$

$$\mathbf{X_k} := \frac{1}{\mathbf{N}} \cdot \sum_{\mathbf{n}=0}^{\mathbf{N}-1} \left(\mathbf{InDat}_{\mathbf{n},1} \cdot \mathbf{e}^{-\mathbf{j}\cdot 2\cdot \pi \cdot \mathbf{k} \cdot \frac{\mathbf{n}}{\mathbf{N}}} \right)$$

This actually performs the complex discrete Fourier transform

In fact, in this case, the argument of the FT was a real one dimensional array of voltage values which was read in. A real array is of course identical to a complex array in which all of the complex coefficients are zero.

Taking an example of the result for the point represented by k = 45, we can see that it is a complex number.

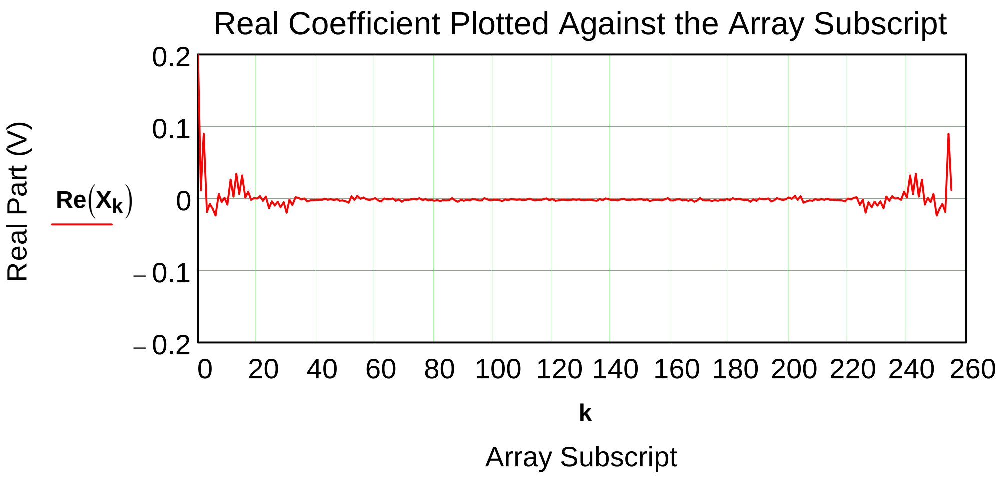
$$\mathbf{X}_{45} = -8.50373168 \times 10^{-4} - 8.09902837j \times 10^{-3}$$

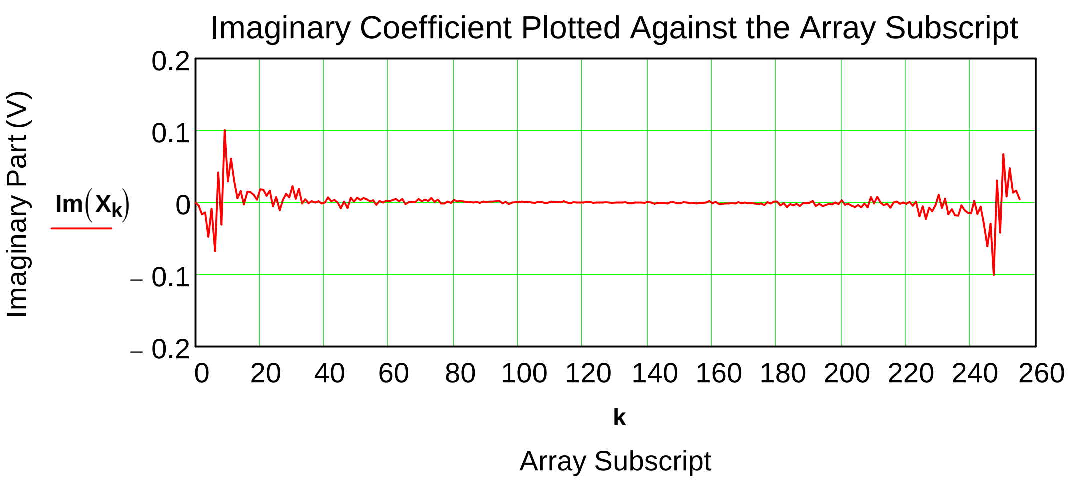
We have an array of complex numbers with a total of N elements and because the subscript starts at zero, it ends at N-1. In this case N=256 so they start at zero and end at 255.

What does this mean?

The following are plots of the real part and the imaginary parts respectively of the result of the complex Fourier transform against the subscript. The units in each case are unchanged from the original voltage against time waveform (volts).

(The Mathcad functions Re(...) and Im(...) mean 'calculate the real part of and calculate the imaginary part of' respectively.)





The real part of the complex numbers represents the coefficient of the cosine waves that go to make up the waveform and the corresponding imaginary parts represent the coefficients of the sine waves.

We can clearly see some symmetry: the right side of each plot is a reflection of its left side. That is because, to represent a sinusoidal waveform using complex coefficients we must always have a positive and a negative part. In this case the positive frequencies are for elements ranging from 0 to N/2 and the negative ones from N/2 to N-1. Whenever we convert to and from the real system we use the positive frequencies. However the negative ones must always be used in the complex calculations for the complex mathematics to work. This results from some manipulation of Euler's equations to give complex exponential expressions for cosine and sine periodic waveforms as follows:

$$\cos \omega t = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right] = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} = \frac{1}{2}e^{j2\pi ft} + \frac{1}{2}e^{-j2\pi ft}$$

$$\sin \omega t = \frac{-j}{2}\left[e^{j\omega t} - e^{-j\omega t}\right] = \frac{-j}{2}e^{j\omega t} + \frac{j}{2}e^{-j\omega t} = \frac{-j}{2}e^{j2\pi ft} + \frac{j}{2}e^{-j2\pi ft}$$

Taking the cosine expression first, it can be seen that this is equivalent to half the sum of the positive exponential plus half the sum of the negative exponential. The exponential constant with the positive power (j2πft) represents a unity magnitude phasor moving in the positive direction (counter-clockwise) and the constant with a negative power (-2πft) represents a similar one moving in a clockwise direction. In each case the frequency of the wave is f. Some consideration of the rotating phasors with their orthogonal components will confirm that this is equivalent to a cosinusoidal oscillation along the real axis.

The sine expression may be analysed in a similar way, but remembering that when a phasor is multiplied by j this is equivalent to rotating it through 90 degrees counter clockwise.

When the result of the Fourier transform is represented graphically, It is usually more convenient to have frequency as the independent variable rather than what can seem a fairly arbitrary subscript, k in this case. Nyquist determined that, for a sample frequency of f_s , the highest frequency that may be reproduced is $f_s/2$. This follows on the basis that one wavelength of a sinusoidal (or cosinusoidal) requires a minimum of 2 samples to be uniquely defined.
We calculated the sampling frequency f_s (in hertz)at the beginning, the reciprocal of the sample spacing. Therefore, the maximum frequency that may be represented is $f_s/2$.
Converting this to the more convenient units of megahertz (MHz), the result must be divided by 10^6, so

$$\mathbf{fmax\_mhz} := \frac{\mathbf{fs}}{2 \cdot 10^6} = 0.5 \quad \mathbf{MHz}$$

The frequency at the subscript N/2 will be fmax_mhz, therefore the frequency step for each subscript step is

$$\mathbf{fstep\_mhz} := \frac{\mathbf{fmax\_mhz}}{\left(\frac{\mathbf{N}}{2}\right)} = 3.90625 \times 10^{-3} \quad \mathbf{MHz}$$

Slightly bizarre frequency steps like this often result when using 2^n type sample points.

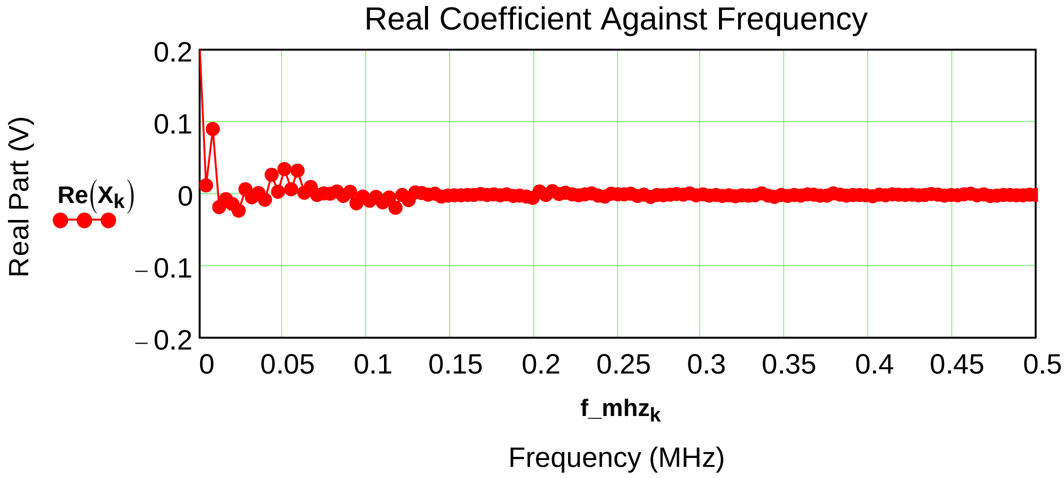
Or if you prefer we can express it in kilohertz

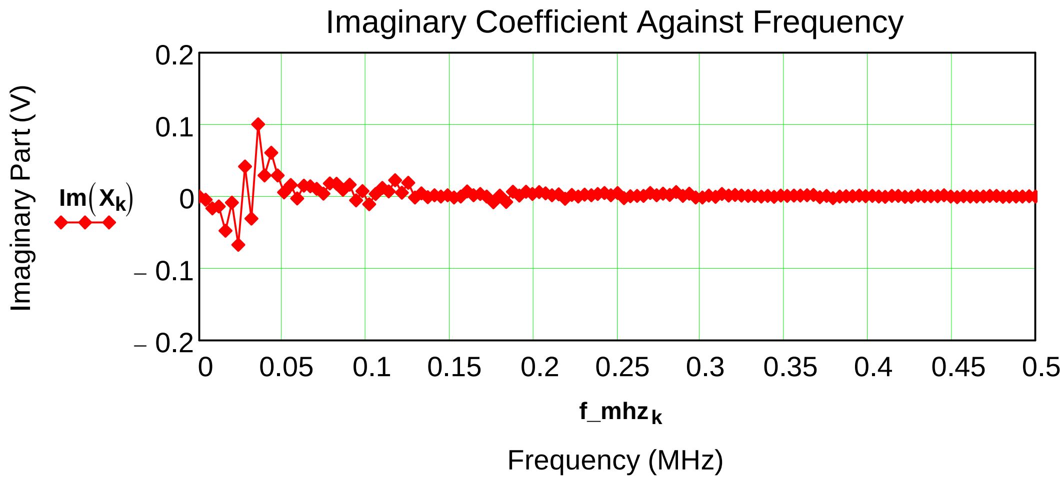
$$\mathbf{fstep\_khz} := \mathbf{fstep\_mhz} \cdot 10^3 = 3.90625 \quad \mathbf{kHz}$$

$\mathbf{f\_mhz_k} := \mathbf{k} \cdot \mathbf{fstep\_mhz}$ Range variable which determinesd the frequency

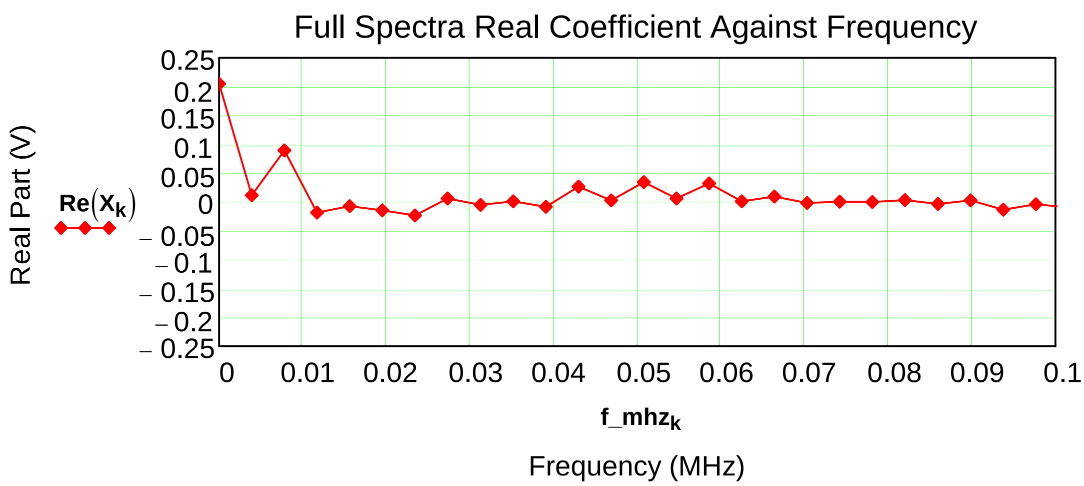
similarly, in kilohertz

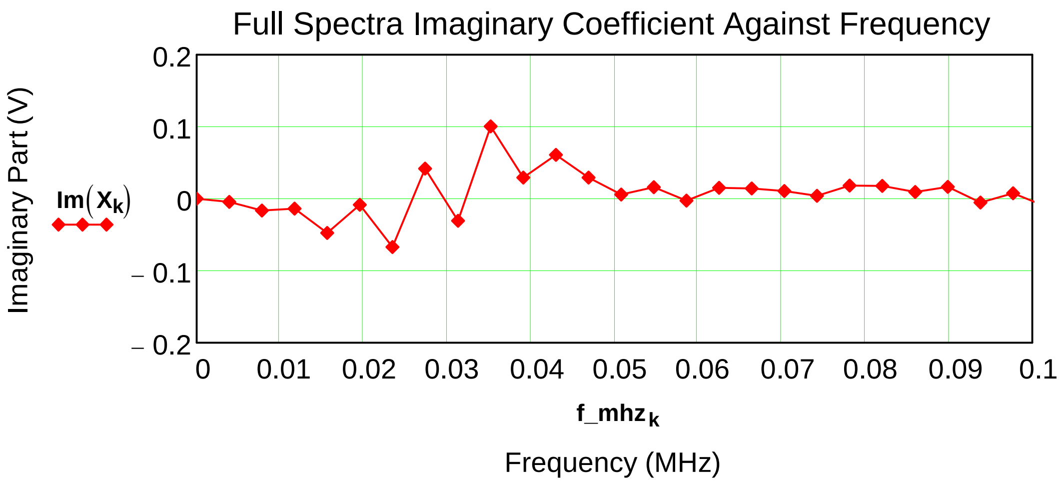
$\mathbf{f\_khz_k} := \mathbf{k} \cdot \mathbf{fstep\_khz}$





In both cases, above about 0.1 MHz the amplitudes are quite small so it might be quite safe to ignore these and concentrate on DC to 0.1 MHz.





Frequency Spectrum

Taking the plots above, we have a set of real coefficients and a set of imaginary coefficients plotted against frequency. Each of these frequencies is made up from orthogonal components: their respective cosine and sine waveforms. Of course the frequency of the orthogonal components must be identical to the frequency under consideration. The coefficient of the cosine waveform is taken from the corresponding real coefficient and that for the sine waveform is taken from the imaginary coefficient.

Lets take an example frequency. The 5th point (k = 4) corresponds to a frequency of

$\mathbf{f\_khz_4} = 15.625$ **kHz**

The value of the complex discrete Fourier transform at this point (X_4) is:

$\mathbf{X_4} = -7.47827304 \times 10^{-3} - 0.04752835j$ **V**

Or alternatively the real and imaginary coefficients at this frequency are

$\mathbf{Re}(\mathbf{X_4}) = -7.47827304 \times 10^{-3}$ **V** ie. the coefficient of the cosine waveform

$\mathbf{Im}(\mathbf{X_4}) = -0.04752835$ **V** ie. the coefficient of the sine waveform

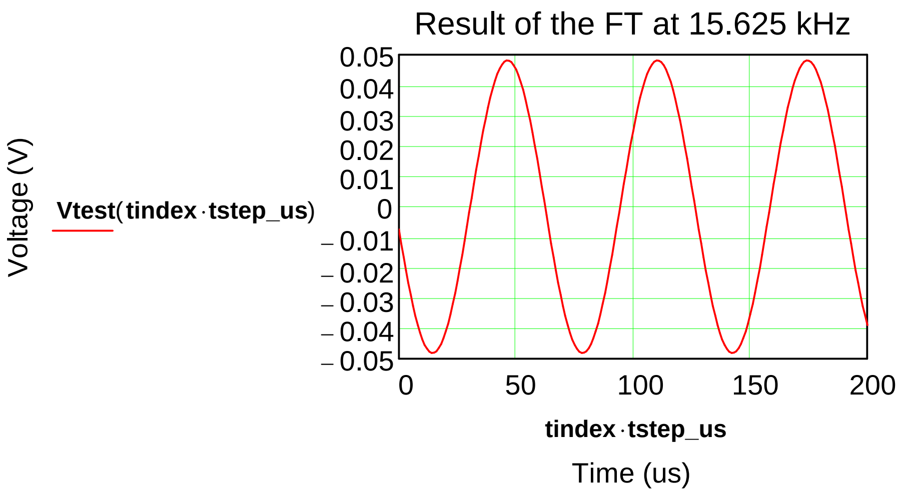
A quick check of k = 4 on the Fourier transform results will confirm that these appear to be about right.

Now we have everything we need to write down the equation for the voltage against time waveform at this frequency. We have to be rather careful with the units of frequency and time as we have changed them a few times. In the function we will use the time in microseconds and the frequency in kilohertz.

$$\mathbf{Vtest(t\_us)} := \mathbf{Re}(\mathbf{X_4})\cdot\mathbf{cos}\left(2\cdot\pi\cdot\mathbf{f\_khz_4}\cdot 10^3\cdot\frac{\mathbf{t\_us}}{10^6}\right) + \mathbf{Im}(\mathbf{X_4})\cdot\mathbf{sin}\left(2\cdot\pi\cdot\mathbf{f\_khz_4}\cdot 10^3\cdot\frac{\mathbf{t\_us}}{10^6}\right)$$

$\mathbf{tindex} := 0 .. 200$

$\mathbf{tstep\_us} := 1$



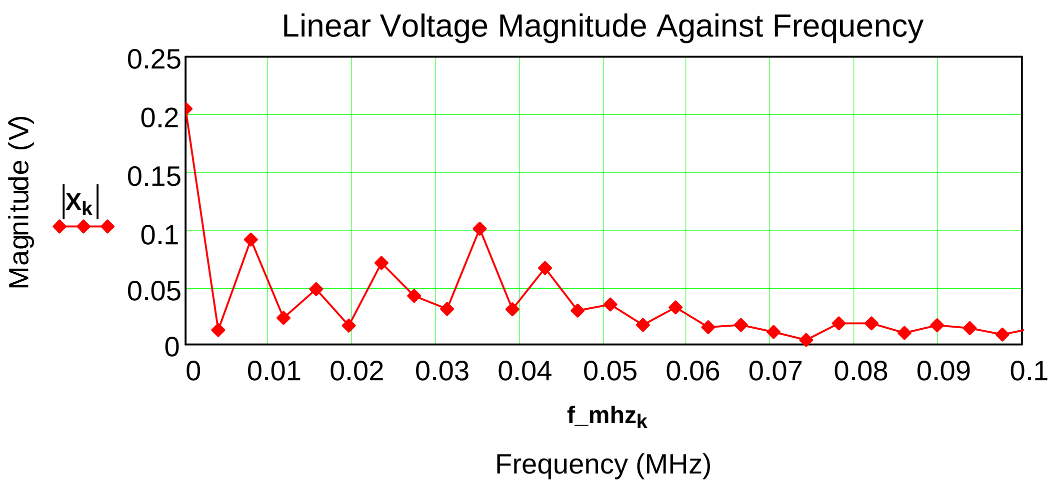
This is equivalent to a sinewave or cosinewave of a particular amplitude and phase.

Units

The units of the FT are the same as the units of the original waveform, in this case volts (V).

Linear Voltage Spectrum

Sometimes it is required to display the linear voltage magnitude against frequency which is also known as the frequency spectrum of the original waveform. This is shown below using the Mathcad absolute value (or magnitude) function.



Logarithmic Magnitude

Often it is required to display the result of the discrete Fourier transform as power levels in absolute logarithmic power units such as decibels relative to one milliwatt (dBm). Typically these units will be used if the spectrum is examined on a spectrum analyzer. To determine the power in this way will require the load impedance, Z0 in Ohms.

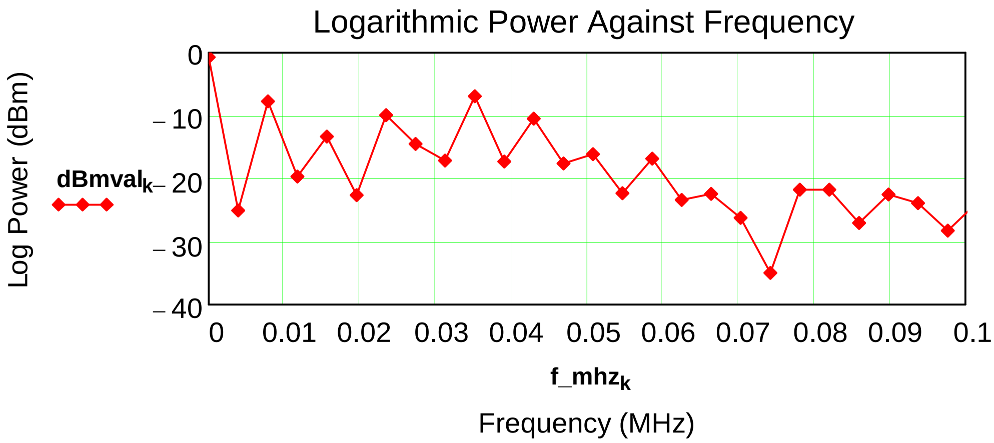
Supposing

Z0 := 50 **Ohms**

Then

$$\mathbf{dBmval_k} := \mathbf{10\,log}\left[10^3\,\frac{\left(\left|\mathbf{X_k}\right|\right)^2}{\mathbf{Z0}}\right]$$ in dBm

This is plotted below



Resolution Bandwidth

In order to display a frequency spectrum like that shown above, the resolution bandwidth of the spectrum analyzer must be set to a sufficiently narrow value such that the power of each point alone is displayed without any additions from adjacent points

Inverse Discrete Complex Fourier Transform

The equation for the inverse discrete complex Fourier transform is

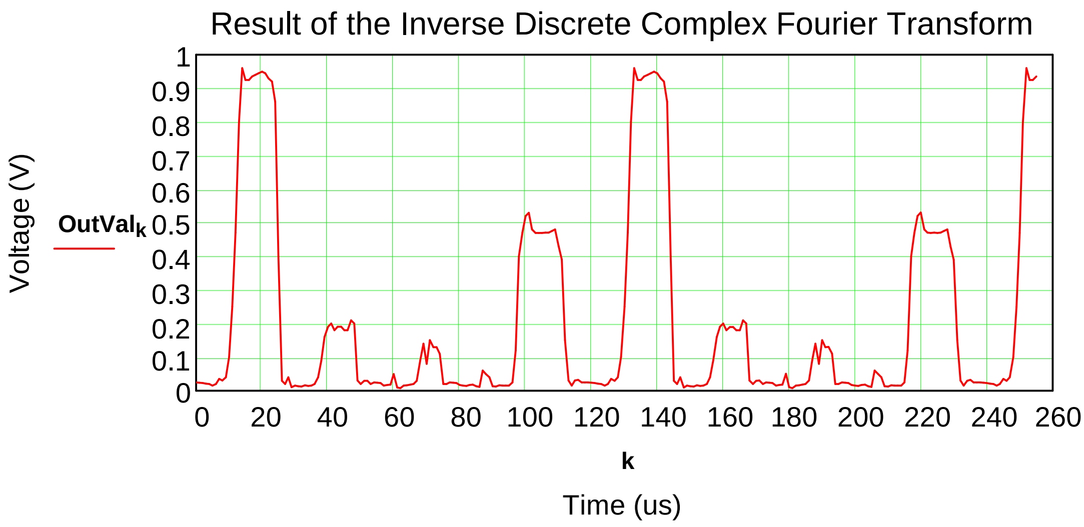
$$x_n = \sum_{m=0}^{N-1} X_m e^{j2\pi mn/N}$$

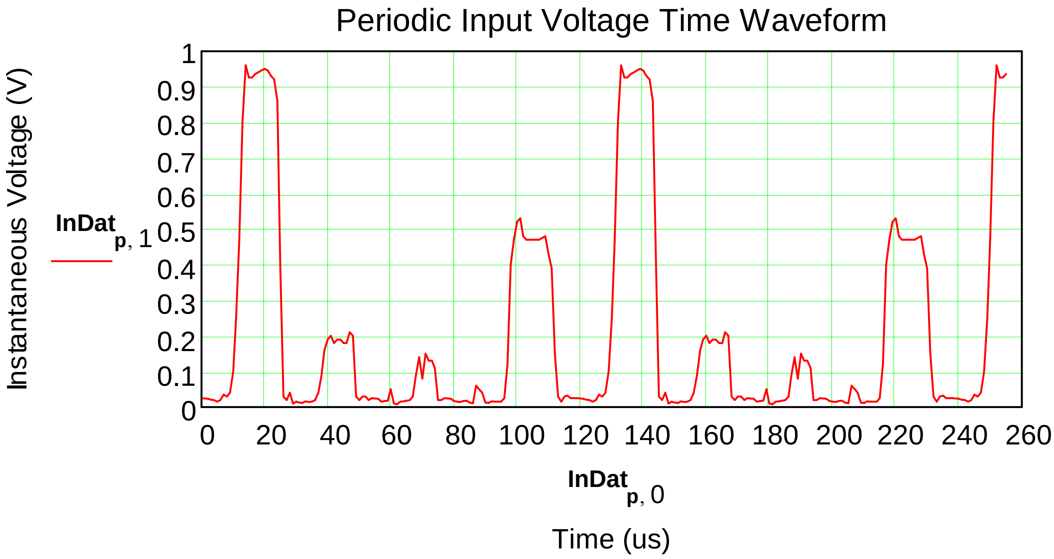
where

- x_n is a discrete variable in the time domain, the result of the inverse transform
- m is the sample number, starting at zero and ending at N-1
- N is the total number of samples taken
- n is the index (subscript) of the (complex) result of the Fourier transform

Using Mathcad we may apply this to the Fourier transform which has already been calculated. If the result is OutVal_k then we should end up with the original voltage against time waveform.

$$\mathbf{OutVal_k} := \sum_{\mathbf{pval}=0}^{\mathbf{N}-1}\left(\mathbf{X_{pval}}\cdot\mathbf{e}^{\mathbf{j}\cdot 2\cdot\pi\cdot\mathbf{pval}\cdot\frac{\mathbf{k}}{\mathbf{N}}}\right)$$





The result of the inverse CDFT is plotted above next to a copy of the original input voltage against time waveform. Clearly there is very good agreement between them.